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Space and Time Evolution of Two Nonlinearly Coupled Variables : Special Solutions

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Further communication about this report is to be sent to the research Information Center, Institute of Plasma Physics, Nagoya University, Nagoya 464, Japan

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Abstract

The system of two coupled linear differential equations are studied assuming that the coupling terms are proportional to the product of the dependent variables, representing e.g. intensities or populations. It is furthermore assumed that these variables experience different linear dissipation or growth. The derivations account for space as well as time dependence of the variables. It is found that certain particular solutions can be obtained to this system, whereas a full solution in space and time as an initial value problem is outside the scope of the present paper. The system has a nonlinear equilibrium solution for which the nonlinear coupling terms balance the terms of linear dissipation. The case of space and time evolution of a small perturbation of the nonlinear equlibrium state, given the initial one-dimensional spatial distribution of the perturbation, is also considered in some detail.

1. Introduction and Basic Equations

Various nonlinear interaction phenomena in plasma physics as well as in modern optics are governed by coupled nonlinear equations of the following form [1,2]

$$\frac{\partial}{\partial\eta} I_0 + \alpha_0 I_0 = s_1 I_0 I_1 \quad , \tag{1}$$

$$\frac{\partial}{\partial\xi} I_1 + \alpha_1 I_1 = s_0 I_0 I_1 \quad , \tag{2}$$

where the variables I_0 and I_1 represent positive quantities, such as intensities or number of quanta, α_0 and α_1 account for linear dissipation of the variables, and s_0 and s_1 denote signs, refering to the effective coupling constants, the absolute amplitudes of which have been normalized to unity. The derivatives are taken with regard to two independent variables.

The system of equations (1) and (2) can be deduced from a set of three equations for interacting waves under the assumption that one of the waves is heavily damped or that the frequency mismatch is large. For this case the third equation can be solved algebraically and one of the variables in the remaining two equations can be eliminated, thus resulting in the set of equations (1) and (2). The quantities I_0 and I_1 denote normalized intensities, defined by

$$I_0 = s_0 \frac{2}{v_1 - v_0} \operatorname{Re}\left(\frac{c_{01}^{*} c_{02}}{\nu_2 - i(\Delta\omega - v_2\Delta k)}\right) |a_0|^2 \quad , \tag{3}$$

$$I_1 = s_1 \frac{2}{v_0 - v_1} \mathrm{Re}\left(\frac{c_{12}^{*} c_{02}}{\nu_2 + i(\Delta\omega - v_2\Delta k)}\right) |a_1|^2 \quad , \tag{4}$$

where a_0 and a_1 are wave amplitudes, c_{ij} coupling coefficients, v_0 and v_1 group velocities, ν_2 and v_2 linear damping and group velocity of the wave which we have formally eliminated, $\Delta\omega$ and Δk frequency and wave-number mismatches, defined as

$$\Delta\omega = \mathrm{Re}(\omega_0) - \mathrm{Re}(\omega_1) - \mathrm{Re}(\omega_2) \quad , \tag{5}$$

$$\Delta k = k_0 - k_1 - k_2 \quad , \tag{6}$$

and where in Eqs.(3) and (4) s_1 and s_2 refer to signs as Eqs.(1) and (2)

The independent variables ξ and η are related to the space and time variables x and t by the relations

$$\xi = x - v_0 t \quad , \tag{5}$$

$$\eta = x - v_1 t \quad , \tag{6}$$

and the coefficients α_0 and α_1 are defined by

$$\alpha_0 = \frac{2\nu_0}{v_0 - v_1} \quad , \tag{7}$$

$$\alpha_1 = \frac{2\nu_1}{v_1 - v_0} \quad , \tag{8}$$

where ν_0 and ν_1 refer to linear dissipation (or growth) of the corresponding waves.

2. Description in Terms of a Potential

As seen from Eqs.(1) and (2) the structure of these equations is such that the evolution of the solutions is governed by the interplay between the nonlinear coupling terms and the remaining linear terms. The signs s_0 and s_1 of the nonlinear terms are therefore essential. Different combinations of signs correspond to different types of solutions and to different physical situations.

When discussing mathematically the solution of Eqs.(1) and (2) it may be convenient, however, to make use of the variables U_i, defined by the relation [2]

$$I_i = s_i U_i \quad . \tag{9}$$

Since in (9) I_i always denotes a positive quantity, we have, however, to make sure that the sign of U_i is such as to correspond to the real physical situation. Eqs.(1) and (2) now take the form[2]

$$\frac{\partial U_0}{\partial \eta} + \alpha_0 U_0 = U_0 U_1 \quad , \tag{10}$$

$$\frac{\partial U_1}{\partial \xi} + \alpha_1 U_1 = U_0 U_1 \quad . \tag{11}$$

It is convenient now to make use of the operator equivalents

$$\frac{\partial}{\partial\eta} + \alpha_0 = e^{-\alpha_0\eta}\frac{\partial}{\partial\eta}\, e^{\alpha_0\eta}, \tag{12}$$

$$\frac{\partial}{\partial\xi} + \alpha_1 = e^{-\alpha_1\xi}\frac{\partial}{\partial\xi}\, e^{\alpha_1\xi}, \tag{13}$$

We then have from Eqs.(10) and (11) that

$$e^{-\alpha_0\eta}\frac{\partial}{\partial\eta}\,(e^{\alpha_0\eta}U_0) = e^{-\alpha_1\xi}\frac{\partial}{\partial\xi}\,(e^{\alpha_1\xi}U_1) = U_0U_1, \tag{14}$$

or accordingly

$$\frac{\partial}{\partial\eta}\,(e^{\alpha_0\eta+\alpha_1\xi}U_0) = \frac{\partial}{\partial\xi}\,(e^{\alpha_0\eta+\alpha_1\xi}\,U_1) = e^{\alpha_0\eta+\alpha_1\xi}U_0U_1 \tag{15}$$

The form (15) suggests the introduction of a potential function $S(\xi,\eta)$, such that

$$e^{\alpha_0\eta+\alpha_1\xi}U_0 = \frac{\partial S}{\partial\xi}\quad, \tag{16}$$

$$e^{\alpha_0\eta+\alpha_1\xi}U_1 = \frac{\partial S}{\partial\eta}\quad. \tag{17}$$

It should be remarked that the potential function S that we have introduced here is related to the function U of reference [2] by $S = e^{X+Y}(1+U)$.

Let us introduce the normalized variables

$$X = \alpha_1\xi\quad, \tag{18}$$

$$Y = \alpha_0\eta\quad. \tag{19}$$

In terms of these variables the equation for S becomes, from Eq.(15)

$$\frac{\partial^2 S}{\partial X \partial Y} = e^{-(X+Y)} \frac{\partial S}{\partial X} \frac{\partial S}{\partial Y} \quad . \tag{20}$$

The functions U_0 and U_1 can then be expressed

$$U_0 = \alpha_1 e^{-(X+Y)} \frac{\partial S}{\partial X} \quad , \tag{21}$$

$$U_1 = \alpha_0 e^{-(X+Y)} \frac{\partial S}{\partial Y} \quad . \tag{22}$$

3. Special Forms of Solutions

Let us try to find solutions of Eq.(20) in terms of the variables

$$Z = X + Y \quad , \tag{23}$$

$$W = X - Y \quad . \tag{24}$$

In Eq.(20) we then have

$$\frac{\partial}{\partial X} = \frac{\partial}{\partial Z} + \frac{\partial}{\partial W} \quad , \tag{25}$$

$$\frac{\partial}{\partial Y} = \frac{\partial}{\partial Z} - \frac{\partial}{\partial W} \quad . \tag{26}$$

and the equation for S can be written

$$\frac{\partial^2 S}{\partial Z^2} - \frac{\partial^2 S}{\partial W^2} = e^{-Z} \{ (\frac{\partial S}{\partial Z})^2 - (\frac{\partial S}{\partial W})^2 \} \quad . \tag{27}$$

In terms of $S(Z,W)$ the functions U_0 and U_1 can be expressed

$$U_0 = \alpha_1 e^{-Z}\left(\frac{\partial S}{\partial Z} + \frac{\partial S}{\partial W} \right) \quad , \tag{28}$$

$$U_1 = \alpha_1 e^{-Z}\left(\frac{\partial S}{\partial Z} - \frac{\partial S}{\partial W} \right) \quad . \tag{29}$$

Before studying explicitly solution which depend on both Z and W let us consider cases where the potential function S depend only on Z.

Case(i): $S = e^Z$, (30)

which satisfies Eq.(27) with $\partial/\partial W \equiv 0$. In this case we have from the expressions (28) and (29), that

$$U_0 = \alpha_1 \quad , \tag{31}$$

$$U_1 = \alpha_0 \quad , \tag{32}$$

which corresponds to the equilibrium situation, where the nonlinear coupling terms balance the linear dissipation terms.

Case(ii): $S = \Phi(\zeta)$, where $\zeta \equiv e^Z$ (33)

In this case we obtain

$$\frac{\partial S}{\partial Z} = e^Z \Phi'(\zeta) \quad , \tag{34}$$

$$\frac{\partial^2 S}{\partial Z^2} = e^Z \Phi'(\zeta) + e^{2Z}\Phi''(\zeta) \quad , \tag{35}$$

and from Eq.(27) with $\partial/\partial W \equiv 0$

$$e^Z \Phi' + e^{2Z}\Phi'' = e^{-Z}(e^Z\Phi')^2$$

which can be written

$$\zeta\Phi'' = (\Phi')^2 - \Phi'$$

or

$$\frac{\Phi}{\Phi'(\Phi'-1)} = \frac{1}{\zeta} \quad . \tag{36}$$

Eq.(36) can be integrated with respect to ζ after first splitting the LHS in two parts

$$\left(\frac{1}{\Phi' - 1} - \frac{1}{\Phi'} \right) \Phi'' = \frac{1}{\zeta} \quad . \tag{36a}$$

From (36a) we then find

$$\ln \frac{\Phi' - 1}{\Phi'} = \ln\zeta + C \quad ,$$

where C is constant, and we can then express Φ' as

$$\Phi' = \frac{1}{1-C\zeta} \quad , \tag{37}$$

and

$$S = \Phi = -\frac{1}{C} \ln|1-C\zeta| + D \ ,$$

where D denotes a constant of integration. From Eqs.(28) and (34) we then have with Eq.(37)

$$U_0 = \frac{\alpha_1}{1-C \exp Z} \quad , \tag{38}$$

$$U_1 = \frac{\alpha_0}{1-C \exp Z} \quad , \tag{39}$$

where $Z = X + Y$.

The solutions (38) and (39) correspond, in fact, to a special case of the solutions found by M. Yamaguchi [4], by means of Hirota's method [5]. Our derivation of this case is however, different and slightly more direct, which may justify to present it here with some further discussion on the solutions.

Expressed explicitly, in space and time variables, we find from relations (9), (18), (19) with (5) and (6) that the normalized intensities I_0 and I_1 can be written, in this case, as

$$I_0 = \frac{\alpha_1 s_0}{1 - C \exp[\alpha_0 (x-v_0 t)+\alpha_1 (x-v_1 t)]} \quad , \tag{40}$$

$$I_1 = \frac{\alpha_0 s_1}{1 - C \exp[\alpha_0 (x-v_0 t)+\alpha_1 (x-v_1 t)]} \quad . \tag{41}$$

Let us analyse briefly some features of the solutions (40) and (41).

Case(ii,I): $0 < C < 1$, ($\alpha_1\ s_0$; $\alpha_0\ s_1$ positive) .

For t=0 we have the initial functions

$$I_0(x,0) = \frac{\alpha_1 s_0}{1-C \exp[(\alpha_0+\alpha_1)x]} \quad , \tag{42}$$

$$I_1(x,0) = \frac{\alpha_0 s_1}{1-C \exp[(\alpha_0+\alpha_1)x]} \quad . \tag{43}$$

In Figure 1 we have indicated the characteristic behaviour of the expressions (42) and (43) by plotting $I_0(x)$.

We notice from Figure 1 and expressions (42) and (43) the explosive type of singularity that may occur.

Introducing the parameter λ, defined by

$$\lambda = \frac{\alpha_0 v_0 + \alpha_1 v_1}{\alpha_0 + \alpha_1} \tag{44}$$

we can write the space and time dependent solutions accordingly

$$I_0(x,t) = \frac{\alpha_1 s_0}{1-C\ \exp[(\alpha_0+\alpha_1)(x-\lambda t)]} \quad , \tag{45}$$

$$I_0(x,t) = \frac{\alpha_0 s_1}{1-C\ \exp[(\alpha_0+\alpha_1)(x-\lambda t)]} \quad , \tag{46}$$

where we notice that

$$x - \lambda t < \frac{1}{\alpha_0+\alpha_1} \ln(1/C) \tag{47}$$

corresponds to the domain of interest in this case. (If on the other hand we consider $\alpha_1 s_0; \alpha_0 s_1$ negative then the domain opposite to that given by (47) should be considered).

Case(ii,II): $C > 1$, ($\alpha_1 s_0$; $\alpha_0 s_1$ positive),

The region of interest in this case becomes

$$x - \lambda t < \frac{-1}{\alpha_0+\alpha_1} \ln C \quad ,$$

or $$t > \frac{1}{\lambda}(x + \frac{1}{\alpha_0+\alpha_1} \ln C) \quad .$$

The solutions tend to $I_0 = \alpha_1 s_0$ and $I_1 = \alpha_0 s_1$, respectively as time tends toward infinity.

In Figure 2 we have plotted the time dependence of I_0 for this case, (for comparison we have also shown the curve for $\alpha_0 s_1$ negative).

Case(ii,III): $C < 0$, ($\alpha_1 s_0 ; \alpha_0 s_1$ positive)

The solutions in this case become

$$I_0 = \frac{\alpha_1 s_0}{1+ C \exp[(\alpha_0+\alpha_1)(x-\lambda t)]} \quad , \tag{48}$$

$$I_1 = \frac{\alpha_0 s_1}{1+ C \exp[(\alpha_0+\alpha_1)(x-\lambda t)]} \quad , \tag{49}$$

with characteristic behaviour as shown in Figure 3. The solutions which we have discussed in Case(ii) have the form of particular stationary solutions in terms of the variable $(x-\lambda t)$.

Let us now consider another case where the solutions depend on Z as well as on W, defined by relations (23) and (24).

Case(iii): $\mathcal{S} = e^Z \Phi(W)$ (50)

From Eq.(27) we obtain

$$e^Z \Phi - e^Z \Phi'' = e^{-Z}[(e^Z \Phi)^2 - (e^Z \Phi')^2] \quad ,$$

or $$\Phi - \Phi'' = \Phi^2 - (\Phi')^2 \quad . \tag{51}$$

Considering case for which $\Phi' \neq 0$ (for $\Phi' = \Phi'' = 0$ we have the equilibrium solution (i)), we notice that by using the identity,

$$[\Phi^2 - (\Phi')^2]' = 2\Phi'(\Phi - \Phi'')$$

combined with relation (51), we obtain

$$\{\ln[\Phi^2 - (\Phi')^2]\}' = 2\Phi' \quad . \tag{52}$$

By integrating Eq.(52) we find

$$\Phi^2 - (\Phi')^2 = A \exp 2\Phi \quad , \tag{53}$$

where A is a constant of integration, or

$$\pm\int_{\Phi_0}^{\Phi} \frac{d\Phi}{\sqrt{\Phi^2 - A\exp(2\Phi)}} = W - W_0 \quad , \tag{54}$$

where $W = W_0$ corresponds to $\Phi = \Phi_0$.

The relation (54) admits oscillatory solutions provided A is positive and small, which is easily seen by writing Eq.(54) in terms of a nonlinear potential

$$V = \frac{1}{2}[\, A\exp(2\Phi) - \Phi^2] \quad . \tag{55}$$

The equation describing the corresponding "particle motion" will be

$$\frac{1}{2}(\Phi')^2 + V = 0 \quad . \tag{56}$$

In figure 4 we have plotted the nonlinear potential (55) indicating that Eq.(56) allows for "particle oscillations" in the negative potential trough $\Phi_0 < \Phi < \Phi_1$ (shaded negion).

For negative values of the constant A we find that $(d\Phi/dW)$ will be a growing function of Φ, and no oscillations will occur.

4. Perturbation of the Nonlinear Equilibrium Solution

As discussed in Case(i) the coupled equations (10) and (11) have equilibrium solutions expressed by Eqs.(31) and (32) as $U_0 = \alpha_1$ and $U_1 = \alpha_0$. Let us perturb this equlibrium and see if we can find oscillatory solutions and how to describe them in a general form.

Let us write Eqs.(10) and (11) in the form

$$\frac{\partial}{\partial \eta} \ln U_0 + \alpha_0 = U_1 \quad , \qquad (57)$$

$$\frac{\partial}{\partial \xi} \ln U_1 + \alpha_1 = U_0 \quad . \qquad (58)$$

By making the change of variable

$$u_1 = U_1/\alpha_0 \quad , \qquad (59)$$

$$u_0 = U_0/\alpha_1 \quad , \qquad (60)$$

and further using the notations X and Y introduced in Eqs.(18) and (19) we can transform Eqs.(57) and (58) into a convenient form for our discussion. In this context we denote the deviations from the equilibria $u_0 = 1$ and $u_1 = 1$ by

$$\Delta_0 = u_0 - 1 \quad , \qquad (61)$$

$$\Delta_1 = u_1 - 1 \quad . \qquad (62)$$

We then obtain the equations

$$\frac{\partial}{\partial Y}[\ln\ (1+\Delta_0)] = \Delta_1 \quad , \qquad (63)$$

$$\frac{\partial}{\partial X}[\ln\,(1+\Delta_1)] = \Delta_0 \quad , \tag{64}$$

We can write Eqs.(63) and (64) for the perturbations in the simple form

$$\frac{\partial \Delta_0}{\partial Y} = \Delta_1(\Delta_0+1) \quad , \tag{65}$$

$$\frac{\partial \Delta_1}{\partial X} = \Delta_0(\Delta_1+1) \quad . \tag{66}$$

Assuming $|\Delta_0| << 1$ and $|\Delta_1| << 1$ we obtain the following linearized equations

$$\frac{\partial \Delta_0}{\partial Y} = \Delta_1 \quad , \tag{67}$$

$$\frac{\partial \Delta_1}{\partial X} = \Delta_0 \quad . \tag{68}$$

We now look for solutions of Eqs.(67) and (68) in the form

$$\Delta_0 = \exp(\, ik_X X - ik_Y Y) \quad , \tag{69}$$

$$\Delta_1 = -ik_Y \exp(\, ik_X X - ik_Y Y) \; , \tag{70}$$

and find from Eq.(68) the condition

$$k_X k_Y = 1 \quad . \tag{71}$$

We therefore find that by using Eqs.(5), (6) and (13), (19) the perturbation Δ_0 can be written explicitly as

$$\Delta_0 = \exp\, i[(k_X\alpha_1 - k_Y\alpha_0)x - (k_X\alpha_1 v_0 - k_Y\alpha_0 v_1)t] \quad , \tag{72}$$

where k_X and k_Y are related according to Eq.(71).

We now assume that initially, at t=0, the perturbation Δ_0 is given by the function G(x), i.e.

$$\Delta_0(x,0) = G(x) = \int e^{iqx} g(q)dq \quad , \tag{73}$$

$$\text{where } g(q) = \frac{1}{2\pi}\int e^{-iqx} G(x)dx \quad , \tag{74}$$

Comparing Eq.(73) with Eq.(72), we notice that, regarding k_X as a Fourier integral variable, q and k_X are related by

$$q = k_X\alpha_1 - \frac{1}{k_X}\alpha_0 \quad , \tag{75}$$

and that generally we have

$$\Delta_0(x,t) = \int \exp[iqx-ik_X(\alpha_1 v_0 - \frac{1}{k_X{}^2}\alpha_0 v_1)t]g(q)dq \quad , \tag{76}$$

with g(q) given by (74).

Eq.(75) has two solutions for k_X (or k_Y), i.e.,

$$\alpha_1 k_X = \{q \pm \sqrt{q^2 + 4\alpha_0\alpha_1}\}/2$$

$$(\text{or } \alpha_0 k_Y = -\{q \mp \sqrt{q^2 + 4\alpha_0\alpha_1}\}/2) \quad ,$$

and when substituted into (72), both satisfy Eq.(67). This indecates that in order to determine the solution of Eq.(67) we need another initial condition. We here assume that

$$[\frac{\partial}{\partial t}\Delta_0(x,t)]_{t=0} = -iH(x) = -i\int e^{iqx} h(q)dq \quad . \tag{77}$$

Writing $\Delta_0(x,t)$ as

$$\Delta_0(x,t)$$
$$= \int \Delta_+(q) \exp i[qx - \{q(v_0+v_1)/2 + \sqrt{q^2+4\alpha_0\alpha_1}(v_0-v_1)/2\}t]dq$$
$$+ \int \Delta_-(q) \exp i[qx - \{q(v_0+v_1)/2 - \sqrt{q^2+4\alpha_0\alpha_1}(v_0-v_1)/2\}t]dq \quad , \tag{78}$$

we determine $\Delta_+(q)$ and $\Delta_-(q)$ from Eqs.(73) and (77) as

$$\Delta_+(q) = \frac{1}{2} g(q) + \frac{h(q)-g(q)q(v_0+v_1)/2}{\sqrt{q^2+4\alpha_0\alpha_1}(v_0-v_1)} \quad , \tag{79}$$

$$\Delta_-(q) = \frac{1}{2} g(q) - \frac{h(q)-g(q)q(v_0+v_1)/2}{\sqrt{q^2+4\alpha_0\alpha_1}(v_0-v_1)} \quad . \tag{80}$$

Introducing (79) and (80) into Eq.(78), we can obtain the complete solution of initial value problem, (67) - (68).

Let us here as a special example consider also the very simple case for which

$\Delta_0(x,0) = G(x) = \Delta_{0c}$, a small constant value, and

$H(x) = \pm\sqrt{\alpha_0\alpha_1}(v_0-v_1)\Delta_{0c}$.

We then have $g(q) = \delta(q)\Delta_{0c}$, $h(q) = \pm\sqrt{\alpha_0\alpha_1}(v_0-v_1)\delta(q)\Delta_{0c}$, and in the integration (76) $k_x = \pm\sqrt{\alpha_0/\alpha_1}$. Therefore

$$\Delta_0(x,t) = \Delta_{0c}\exp[i\pm\sqrt{\alpha_0\alpha_1}(v_0-v_1)t]$$

Remembering the relations (7) and (8) we obtain

$$\Delta_0(x,t) = \Delta_{0c}\exp[\pm 2i\sqrt{(-1)\nu_1\nu_2}t] \quad . \tag{81}$$

From expression (81) we notice that if the linear dissipation coefficients ν_1 and ν_2 have different signs there will be oscillations — relaxation oscillations — of the small initial perturbation around the equibrium level.

5. Concluding remarks

We have here given brief discussions of some special solutions of the equations for the nonlinearly coupled variables, when these vary in space and time. The coupled equations, which in the general form describe so many and interesting physical phenomena are indeed challenging to study and a complete solution would be desirable. However, since we are lacking such a solutions we believe that examples of special solutions may shed some light on the properties of these equations and on the phenomena that they describe.

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Figure Captions

Fig. 1. Qualitative plot of the initial dependence of the quantity I_0 on x for $0 < C < 1$.

Fig. 2. Qualitative plot of I_0 as a function of t for $C > 1$

Fig. 3. Qualitative plot of I_0 as a function of t for $C < 0$

Fig. 4. Qualitative plot of the nonlinear potential (55).

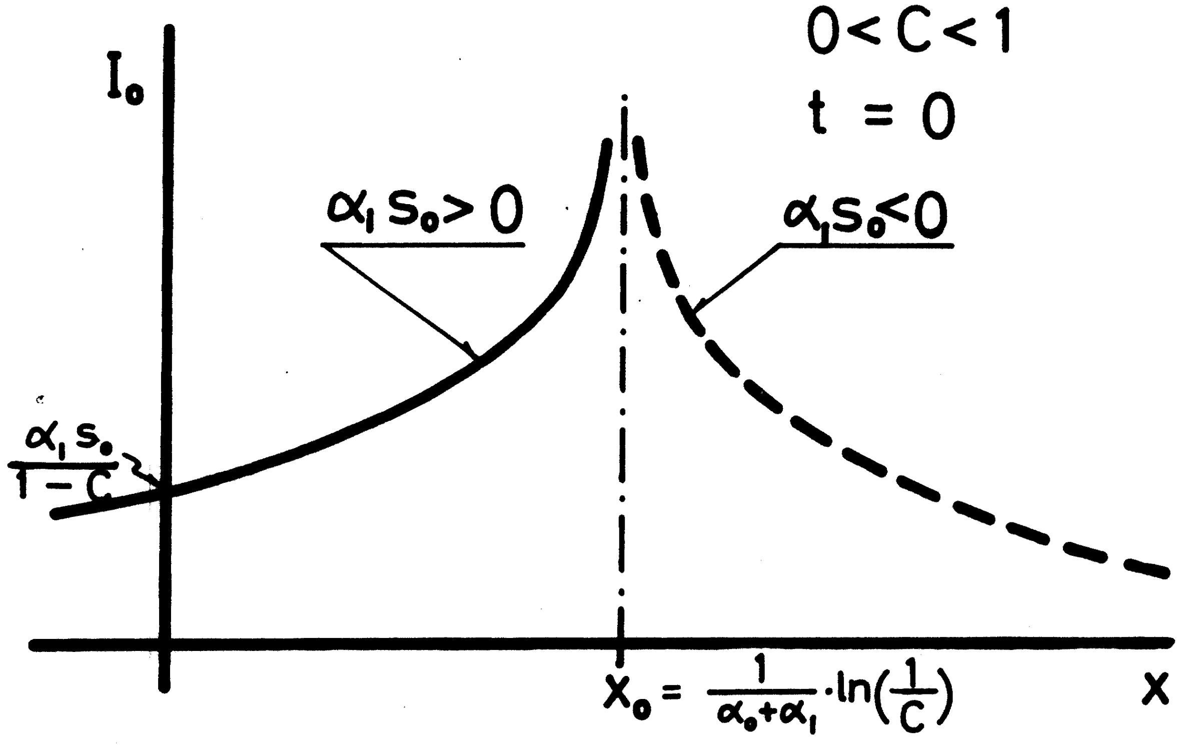
0<C<1
t = 0
I_0
$\alpha_1 s_0 > 0$
$\alpha_1 s_0 < 0$
$\frac{\alpha_1 s_0}{1-C}$
$x_0 = \frac{1}{\alpha_0+\alpha_1} \cdot \ln(\frac{1}{C})$
x

FIG. 1

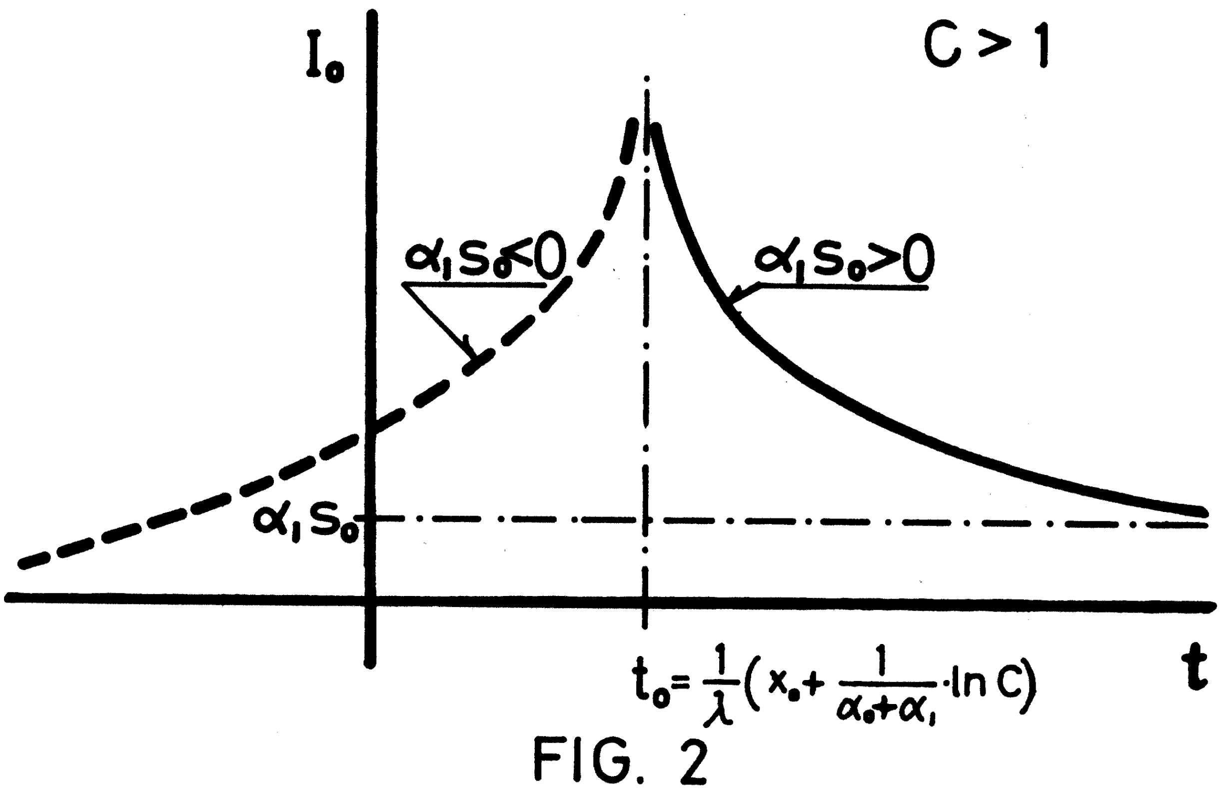
C > 1
I_0
$\alpha_1 s_0 < 0$
$\alpha_1 s_0 > 0$
$\alpha_1 s_0$
$t_0 = \frac{1}{\lambda}(x_0 + \frac{1}{\alpha_0+\alpha_1} \cdot \ln C)$
t

FIG. 2

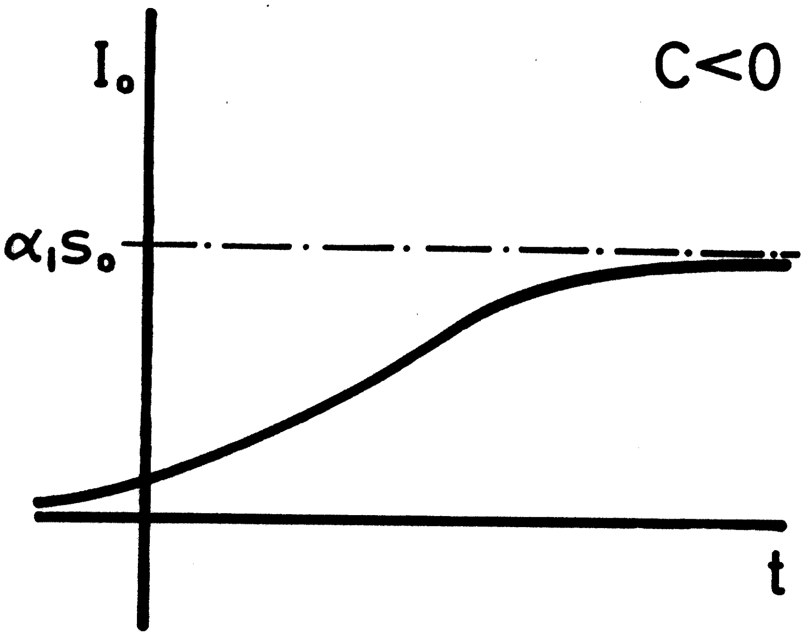

FIG. 3

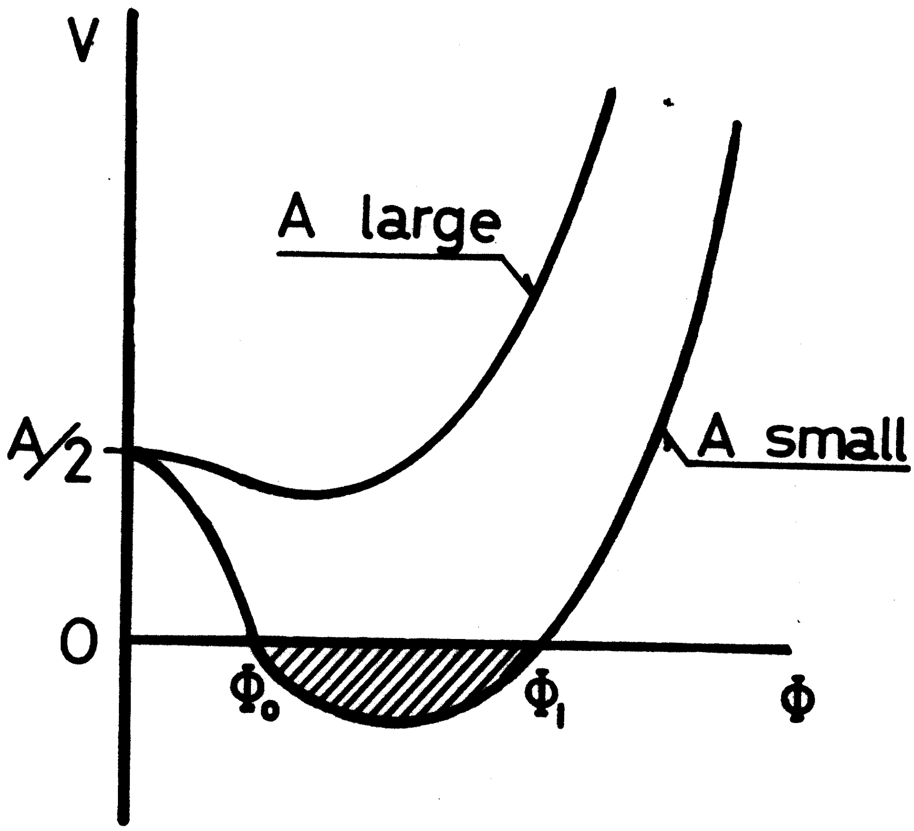

FIG. 4