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Theory for Space-Time Evolution of Explosive-type Instability

M. Watanabe*, Kyoji Nishikawa*

and

H. Wilhelmsson**

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Permanent address :

* Faculty of Science, Hiroshima University, Hiroshima, Japan.

** JSPS Visiting Professor, on leave of absence from the Institute for Electromagnetic Field Theory, Chalmers University of Technology, S-40220 Göteborg, Sweden.

Abstract

We present a solution describing space-time evolution of a resonantly interacting system of a damped negative energy wave and two undamped wave packets of positive energies. It is shown that all three waves grow simultaneously when they are approaching each other, while a certain threshold condition is required for instability when they are separating.

It is well-known that when a negative energy wave couples resonantly with two lower-frequency waves, all three waves grow simultaneously resulting in an explosion of their amplitudes in a finite time scale [1]. In most previous works [2-5], evolution of this explosive instability has been treated as a function of only one independent variable, space or time. Space-time evolution was first studied by Zakharov and Manakov [6] by the inverse scattering method. This method has been extended in Refs.[7] and [8] to treat a more concrete situation where a large-amplitude negative energy wave interacts with two small-amplitude positive energy wave packets. In this paper, we consider the situation where the negative energy wave is damped in the absence of coupling, but is nonlinearly created by two positive energy waves with localized initial space distribution of intensities. We use a method similar to that of Cohen [9], which treated a stable case, i.e. an interaction of three positive energy waves. We show that the interaction always results in growth of all three waves when they are approaching each other, whereas when they are separating instability requires a certain threshold condition which is determined by the damping of the negative energy mode and the convection loss of the positive energy modes.

We limit our study to a one-dimensional system in a homogeneous medium and to the resonant explosive type of interaction for which the wavenumber and frequency matching conditions, $k_0=k_1+k_2$ and $\omega_0=\omega_1+\omega_2$, are satisfied, where the suffix o refers to the negative energy mode. We then investigate the slow space-time evolution of the envelopes $a_i(i=0, 1, 2)$ having the group velocities V_i. By taking into account the damping Γ for the negative energy mode alone and by making the transformation $\zeta = x - V_0 t$, we then have the following set of coupled equations,

$$\frac{\partial}{\partial t} a_0 + \Gamma a_0 = \beta a_1 a_2 \tag{1}$$

$$\frac{\partial}{\partial t} a_1 + \tilde{V}_1 \frac{\partial}{\partial \zeta} a_1 = \beta a_1 a_2^* \tag{2}$$

$$\frac{\partial}{\partial t} a_2 + \tilde{V}_2 \frac{\partial}{\partial \zeta} a_2 = \beta a_0 a_1^* , \tag{3}$$

where $\tilde{V}_i = V_i - V_0 (i=1,2)$ and we have normalized the amplitudes such that all equations have the same coupling constant β.

Without loss of generality, we can write the envelope of the negative energy mode in the following form:

$$a_0(\zeta,t) = A(\zeta)\exp[\int_0^t \lambda(\zeta,t')dt'] \tag{4}$$

$$= [\beta/(\lambda+\Gamma)]a_1 a_2 , \tag{5}$$

where in (5) we used Eq.(1). Introducing the intensities defined by $I_i = |a_i|^2 (i=0,1,2)$, we then have from (5)

$$I_0 = [|\beta|^2/(\lambda+\Gamma)^2] I_1 I_2 , \tag{6}$$

and by furthermore introducing the variables, $\xi = \zeta - \tilde{V}_1 t$ and $\eta = \zeta - \tilde{V}_2 t$, we obtain from (2) and (3),

$$\frac{\partial I_1}{\partial \eta} = \mu I_1 I_2 , \quad \frac{\partial I_2}{\partial \xi} = -\mu I_1 I_2 , \tag{7}$$

where

$$\mu = 2|\beta|^2/[(\lambda+\Gamma)(\tilde{V}_1 - \tilde{V}_2)] . \tag{8}$$

We solve Eqs.(7) by noting that the solution can be generated from a potential $\Phi(\xi,\eta)$ as

$$I_1(\xi,\eta) = \frac{\partial}{\partial\xi}\Phi(\xi,\eta), \quad I_2(\xi,\eta) = -\frac{\partial}{\partial\eta}\Phi(\xi,\eta) , \quad (9)$$

and by restricting ourselves to the case of small growth rate, i.e. $|\lambda| << \Gamma$, which is always satisfied near threshold. In this case, we can safely replace λ in (8) by an average value $\overline{\lambda}$ which is independent of ξ and η. The solution can then readily be obtained in the form [10]

$$\Phi(\xi,\eta) = \mu^{-1} \log [f(\xi) + g(\eta)] \quad (10)$$

$$I_1(\xi,\eta) = \mu^{-1} f'(\xi) / [f(\xi) + g(\eta)] \quad (11)$$

$$I_2(\xi,\eta) = -\mu^{-1} g'(\eta) / [f(\xi) + g(\eta)] , \quad (12)$$

where f and g are, respectively, functions of ξ and η alone and can be determined from the initial conditions as follows.

Let the initial spatial distribution of I_i be $P_i(x)$ $(i=1,2)$. Since $t=0$ corresponds to $\xi=\eta$, we have

$$I_i(\xi,\xi) = P_i(\xi) \qquad (i=1,2) . \quad (13)$$

Then noting the relation $\mu[P_1(\xi)-P_2(\xi)]=(d/d\xi)\log[f(\xi)+g(\xi)]$, which follows from (11) and (12), we obtain the following expressions for f and g,

$$f(\xi) = \mu\int_{\xi_0}^{\xi} P_1(\xi')F(\xi')d\xi' + 1 + C \quad (14)$$

$$g(\xi) = -\mu\int_{\xi_0}^{\xi} P_2(\xi')F(\xi')d\xi' - C , \quad (15)$$

where C is an arbitrary constant and

$$F(\xi) \equiv f(\xi)+g(\xi) = \exp[\mu\int_{\xi_0}^{\xi} \{P_1(\xi')-P_2(\xi')\}d\xi']. \quad (16)$$

Our final task is to determine the growth rate $\lambda(\xi,\eta)$ self-consistently. To this end, we assume that $P_i (i=1,2)$ are sufficiently small in the sense that the integrals of P_i from ξ_0 to ξ or η are small as compared to μ^{-1}. Then to lowest order, one can approximate f and g by 1+C and -C and f' and g' by μP_1 and $-\mu P_2$, respectively. This approximation yields a solution which describes free propagation of the two positive energy wave packets,

$$I_1^{(0)}(\xi,\eta) = P_1(\xi), \quad I_2^{(0)}(\xi,\eta) = P_2(\eta) , \qquad (17)$$

where the superscript (0) depicts the lowest order approximation. In the next order, we treat the integrals of P_i as perturbations and obtain

$$f(\xi) = 1+C+\mu\int_{\xi_0}^{\xi} P_1(\xi')d\xi', \quad g(\eta) = -C-\mu\int_{\xi_0}^{\eta} P_2(\xi')d\xi'$$

$$f'(\xi) = \mu P_1(\xi)\{1+\mu\int_{\xi_0}^{\xi} [P_1(\xi')-P_2(\xi')]d\xi'\}$$

$$g'(\eta) = -\mu P_2(\eta)\{1+\mu\int_{\xi_0}^{\eta} [P_1(\xi')-P_2(\xi')]d\xi'\} ,$$

from which we have

$$I_1(\xi,\eta) = P_1(\xi)\{1+\mu\int_{\xi}^{\eta} P_2(\xi')d\xi'\} \qquad (18)$$

$$I_2(\xi,\eta) = P_2(\eta)\{1+\mu\int_{\xi}^{\eta} P_1(\xi')d\xi'\} . \qquad (19)$$

Substituting (18) and (19) into (6) where again we replace $(\lambda+\Gamma)$ by $(\overline{\lambda}+\Gamma)$, and comparing the result with (4), we obtain

$$2\lambda(\zeta,t) = \{\frac{\partial}{\partial t} \log[I_1(\xi,\eta)I_2(\xi,\eta)]\}_{\zeta=\mathrm{const.}}$$

$$= \mu\{\tilde{V}_1[P_1(\xi)+P_2(\xi)] - \tilde{V}_2[P_1(\eta)+P_2(\eta)]\}$$

$$- \tilde{V}_1 P'_1(\xi)/P_1(\xi) - \tilde{V}_2 P'_2(\eta)/P_2(\eta) \quad . \tag{20}$$

The first term in Eq.(20), i.e. the term proportional to the coupling μ, is the contribution to the growth of the negative energy mode due to the three-mode coupling; note that since μ is proportional to $(\tilde{V}_1-\tilde{V}_2)^{-1}$, the coupling always contributes to growth provided $|\tilde{V}_1-\tilde{V}_2|$ is much greater than $|\tilde{V}_i P_i|$. The last two terms in Eq.(20) describe the effect of convection of the two positive energy modes relative to the negative energy mode. One can alternatively regard the growth rate (20) as consisting of the sum of the growth rate of mode 1, which is $\mu[\tilde{V}_1 P_1(\xi)-\tilde{V}_2 P_1(\eta)]$ $-\tilde{V}_1 P'_1(\xi)/P_1(\xi)$, and that of mode 2, which is $\mu[\tilde{V}_1 P_2(\xi)-\tilde{V}_2 P_2(\eta)]$ $-\tilde{V}_2 P'_2(\eta)/P_2(\eta)$. We note that $V_i P_i'$ is negative or positive depending on whether the mode i is approaching or departing the point under consideration. This implies that if the negative energy mode is also localized in its wave frame, all three modes have positive growth rate when they are approaching each other, while when they are separating the growth of the negative energy mode requires a threshold condition to be fulfilled. In particular, the condition for initial growth at the origin, i.e., at $\xi=\eta=0$, is given by

$$P_1(0) + P_2(0) > (\Gamma/2|\beta|^2)(K_1\tilde{V}_1+K_2\tilde{V}_2) \ , \tag{21}$$

where $K_i=[P'_i(x)/P_i(x)]_{x=0}$.

To illustrate the result, let us consider the case where $P_i(x)$ are given by

$$P_1(x) = J_1 \mathrm{sech}^2 K(x+x_0), \quad P_2(x) = J_2 \mathrm{sech}^2 K(x-x_0). \tag{22}$$

In this case, our perturbation analysis is valid when $(J_1+J_2)|\mu/K|$ <<1. The initial growth rate in this case is given be

$$\lambda = \frac{|\beta|^2}{\Gamma+\bar{\lambda}} [J_1 \mathrm{sech}^2 K(x+x_0) + J_2 \mathrm{sech}^2 K(x-x_0)]$$
$$+ 2K[\tilde{V}_1 \mathrm{th} K(x+x_0) + \tilde{V}_2 \mathrm{th} K(x-x_0)] . \tag{23}$$

For $\tilde{V}_1$, $\tilde{V}_2>o$, the convection contribution is positive (of order $2K(\tilde{V}_1+\tilde{V}_2)$) at $x\to\infty$ and negative at $x\to-\infty$. The interaction term gives a positive contribution near $x=\pm x_0$. The maximum growth rate is obtained when Kx_0 is of order unity. The condition $|\lambda|<<\Gamma$ is satisfied as far as

$$(|\beta|^2/\Gamma)(J_1+J_2) + |K\tilde{V}_1|+|K\tilde{V}_2| << \Gamma \quad . \tag{24}$$

This same condition guarantees the inequality $|\lambda|<<\Gamma$ at all times provided $|(V_1-V_2)/V_i|$ is of order unity.

As a final remark, we note that although we have assumed homogeneity of the medium, the present result can equally be applied to an inhomogeneous medium if the group velocity V_0 is equal to zero. As is well-known [11], inhomogeneity yields a phase factor $\exp[-i\int^{\zeta}\kappa d\zeta']$ on the right-hand side of Eq.(1) and $\exp[i\int^{\zeta}\kappa d\zeta']$ on those of Eqs.(2) and (3), where $\kappa(=k_0(\zeta)-k_1(\zeta)-k_2(\zeta))$ is the wavenumber mismatch due to the inhomogeneity. If $V_0=0$, however, these phase factors just cancel out in Eqs.(6) and (7), so that the above result is unchanged by the wavenumber mismatch. Physically, this is due to the fact that the negative energy mode has a frequency independent of the wavenumber and hence for given frequencies, $\omega_1+\omega_2=\omega_0$, one can always adjust the value of k_0 so that the wavenumber matching is satisfied at all positions.

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