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Theory of two-fluid equilibrium of an axially symmetric plasma

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ABSTRACT

New formulation for determining the two-fluid equilibrium of an axially symmetric plasma is presented. The inertia force due to plasma motion is taken into account exactly in the form of a modified electrostatic potential (due to the centrifugal force) and a modified magnetic field (due to Coriolis' force). A set of partial differential equations which determine the modified flux functions and the self-consistent electric and magnetic fields are derived under the assumption that the pressure is isotropic and is known as a function of the density and the flux function from an equation of state. In addition, the poloidal drift flux on the modified magnetic surface and the density and temperature in the absence of both the electrostatic potential and the plasma flow are assumed to be known. Our formulation is particularly useful for the case when a local charge separation causes a deviation of the equipotential surface from the magnetic surface, as has been observed in several recent experiments. The result is also formulated in the form of a variation principle which contains no constraints and hence is useful for numerical and perturbational analysis.

I. INTRODUCTION

Toroidal equilibrium of a plasma has extensively been studied within the framework of the ideal magnetohydrodynamic (MHD) theory. For an axially symmetric plasma, the equilibrium state is determined by the set of equations[1]

$$\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} = -16\pi^3 r^2\left\{\frac{dP}{d\psi} + \frac{1}{8\pi r^2}\frac{d}{d\psi}(r\,B_\phi)^2\right\}$$

$$P = P(\psi)\ , \qquad rB_\phi = 2I(\psi)/c\ , \tag{1}$$

where we used the cylindrical coordinate system, (r, ϕ, z), $\psi(=2\pi rA_\phi)$ is the flux function, P and I are, respectively, the plasma kinetic pressure and the axial current which are arbitrary functions of ψ, and B_ϕ, A_ϕ are the ϕ-components of the magnetic field and the vector potential, respectively.

Applicability of this ideal MHD theory is, however, limited on the following three points: 1) the inertia force due to the plasma motion is neglected; 2) the electric field inside the plasma is left undetermined and its effect on the plasma motion is taken into account only through the $\vec{E}\times\vec{B}$ drift; and 3) generalization to multi-component plasma is not straightforward.

Effects of the inertia force as well as the gravitational force on the one-fluid MHD equilibrium were first studied by Woltijer[2]. He showed that the MHD equilibrium is determined by a second-order partial differential equation whose structure becomes either of elliptic type or of parabollic type depending on the value of the flow velocity. Dobrott and Greene[3] have examined the stationary state of an axially symmetric plasma using the collisionless MHD equations with anisotropic pressure tensor together with the equations for the guiding centers of particles, and have derived a second order partial differential equation with the aid of the macroscopic and the microscopic constants of motion. In particular, they noted the possibility of having a weak solution (shock solution) in the parabolic regime, in which case he derived the jump condition.

Effects of plasma flow have also received considerable attention in connection with an anomalous increase of the plasma diffusion[4,5,6]. In particular, several authors[7-10] have pointed out a formation of steep density gradient when the plasma rotational velocity approaches a certain critical value. To the author's knowledge, these works have been restricted to the investigation within the framework of the one-fluid MHD theory.

Now, a steep density gradient can cause a charge separation in a plasma, which invalidates the one-fluid MHD equations. In particulars, the equipotential surface can deviate from the magnetic surface, whereas in the one-fluid MHD theory they are identical to each other. Recent experimental result[11] for the JIPP-I Stellarater indicates the existence of a convection flow of the plasma and the associated deviation of the magnetic surface from the equipotential surface. Leung et al.[12] have observed a stationary state in a picket fence configuration where the electrostatic potential varies along the magnetic line of force.

In this paper, we present a theory for an axially symmetric plasma equilibrium based on the collisionless two-fluid equations. Our formulation includes the electron and the ion inertia forces, and can describe the situation where the charge separation yields an equipotential surface different from the magnetic surface. The basic physical quantities which we assume to be given are 1) the distribution of the poloidal particle flux as a function of a modified magnetic flux function (modification due to Coriolis' force of the plasma flow), and 2) the density and temperature distributions in the absence of the electrostatic potential and the plasma flow. Knowing these quantities and assuming an equation of state which relates the plasma pressure to the plasma density and temperature, we can determine all the other fluid and electromagnetic variables as solutions of a set of partial differential equations.

The method to be presented in this paper can readily be generalized to a multi-component plasma. It can also be generalized to a helically symmetric plasma. Although for simplicity we assume an isotropic plasma pressure, generalization to the case of an anisotropic pressure tensor is also possible. These generalizations will be presented in a separate paper.

The present formulation of the two-fluid plasma equilibrium can also be phrased in the form of a variation principle. Greene and Karlson[13] have derived a variation principle for the one-fluid MHD equilibrium. Whereas this variation principle is to be solved subject to some additional constraints, the two-fluid variation principle is free from the constraints and hence directly determines the flux function.

In Sec. II, we present the basic two-fluid equations where the inertia force is divided into the centrifugal force and Coriolis' force, these modifying the electric and the magnetic fields respectively. In Sec. III, we derive the equations which determine the plasma flow and density when the electric and the magnetic fields are given. The latter are determined from the Maxwell and Poisson set of equations in Sec. IV where the final set of partial differential equations which determine the flux functions are also given. Sec. V is devoted to the derivation of a variation principle which is equivalent to the partial differential equations derived in Sec. IV. A brief conclusion is added in the last Section. Two Appendices deal with a similar formulation of the ideal MHD equilibrium and some specific examples for the two fluid equilibrium.

II. BASIC EQUATIONS

We consider a plasma described by the electron and the ion fluids with no mutual exchange of energy and momentum. For simplicity, we assume an isotropic partial pressure, P_e and P_i, although generalization to anisotropic cases can readily be made.

The two-fluid equations for the force balance and the mass conservation along with the coupled Maxwell and Poisson equations can be written in the form:

$$(\vec{v}_j \cdot \vec{\nabla})\vec{v}_j = \frac{q_j}{m_j} (-\vec{\nabla}\Psi + \frac{1}{c} \vec{v}_j \times \vec{B}) - \frac{1}{m_j n_j} \vec{\nabla} P_j \tag{2}$$

$$0 = \vec{\nabla}\cdot(n_j \vec{v}_j) \tag{3}$$

$$- \nabla^2 \Psi = 4\pi \sum_{j=e,i} q_j n_j \tag{4}$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \sum_{j=e,i} q_j n_j \vec{v}_j \tag{5}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{6}$$

where $\vec{v}_j$, n_j, q_j and m_j are, respectively, the fluid velocity, density, charge and mass of the j-th species of particle with j denoting either the electron (j=e) or the ion (j=i), Ψ denotes the electrostatic potential and $\vec{B}$ the magnetic field.

In Eq.(2), the inertia force $(-\vec{v}_j \cdot \vec{\nabla})\vec{v}_j$, can be divided into two parts: centrifugal force, $-\vec{\nabla}|v_j|^2/2$, and Coriolis' force, $\vec{v}_j \times (\vec{\nabla} \times \vec{v}_j)$. The centrifugal force is derived from a scalor potential which can be incorporated into the electrostatic potential as

$$\chi_j \equiv \Psi + \frac{m_j}{q_j} \frac{1}{2} |\vec{v}_j|^2 . \tag{7}$$

We refer to χ_j is the modified electrostatic potential. Coriolis' force has the same tensorial character as the Lorentz force due to the magnetic field and hence can be taken into account by introduction of the modified magnetic field defined by

$$\vec{\Omega}_j = \vec{B} + \frac{m_j}{q_j} c \vec{\nabla} \times \vec{v}_j . \tag{8}$$

The modified magnetic field satisfies the same relation as Eq.(6) for the real magnetic field, i.e.

$$\vec{\nabla}\cdot\vec{\Omega}_j = 0 \quad . \tag{9}$$

This equation will be used in place of Eq.(6).

Using these modified electrostatic potential and magnetic field, we can write Eq.(2) as

$$0 = -q_j\vec{\nabla}\chi_j + \frac{q_j}{c}\vec{v}_j \times \vec{\Omega}_j - \frac{1}{n_j}\vec{\nabla} P_j \quad . \tag{10}$$

In accordance with the modification of the magnetic field, the flux function is also to be modified. For an axially symmetric plasma, which we consider in this paper, the modified flux function, $\psi_j(r,z)$, is defined by the relation

$$\vec{\Omega}_j = \left(-\frac{1}{2\pi r}\frac{\partial\psi_j}{\partial z}, \quad \Omega_{j\phi}, \quad \frac{1}{2\pi r}\frac{\partial\psi_j}{\partial r}\right) , \tag{11}$$

where we used the cylindrical coordinate system, (r,ϕ, z).

To complete the basic equations, we need a relation which determines the plasma pressure P_j in terms of the other fluid and electromagnetic variables. Such a relation can be provided if we assume an equation of state which relates the pressure to the density. In the two-fluid equilibrium, the isobaric surface, $P_j(r,z)$=const., and the modified magnetic surface, $\psi_j(r,z)$ = const., are in general different from each other, i.e.,

$$\frac{\partial(P_j, \psi_j)}{\partial(r, z)} \neq 0 \quad . \tag{12}$$

We can then use P_j and ψ_j as independent variables instead of r and z. The equation of state will then yield the following form of relation,

$$n_j = n_j(P_j, \psi_j) \quad . \tag{13}$$

The simplest form of such relation is, $n_j = P_j/T_j(\psi_j)$, where T_j is the temperature.

Our basic equations are then Eqs.(3), (4), (5), (9) and (10) supplemented by the relations (7), (8), (11) and (13).

III. DETERMINATION OF FLUID VARIABLES

In this Section, we derive the equations which determine the fluid variables in terms of the electric and the magnetic fields. To simplify the notation, we shall suppress suffix j throughout this Section.

We start from the ϕ-component of the force balance equation (10). Noting that all quantities are independent of ϕ, we find the relation

$$0 = \frac{1}{2\pi r}\frac{q}{c}\vec{v}\cdot\vec{\nabla}\psi \quad ,$$

which admits us to write $\vec{v}$ in the form

$$\vec{v} = (-h(r,z)\frac{\partial\psi}{\partial z} \, , \; v_\phi, h(r,z)\frac{\partial\psi}{\partial r}) \quad . \tag{14}$$

The function h(r,z) can be determined from the equation of continuity (3) as follows:

$$0 = \vec{\nabla}\cdot(n\vec{v}) = \frac{1}{r}\frac{\partial(\psi, nrh)}{\partial(r, z)}$$

or

$$h(r, z) = F(\psi)/nr \, , \tag{15}$$

where $F(\psi)$ is an arbitrary function of ψ. This function specifies the poloidal drift flux of particles on the modified magnetic surface and is determined by the balance of the production rate and the diffusion rate of the plasma. In this paper, we are neglecting these effects, whence we simply assume that $F(\psi)$ is a known function. Then, since the density n is assumed to be known by the equation of state (13), the r, z-components of the fluid velocity are determined in terms of P and ψ.

The r, z - components of the force balance equation (10) can now be written as,

$$\frac{q}{c}\left(\frac{V_\phi}{2\pi r} - \frac{F}{nr}\Omega_\phi\right)\vec{\nabla}\psi - q\vec{\nabla}\chi - \frac{1}{n}\vec{\nabla} P = 0. \tag{16}$$

Noting the relation (12), we express χ as a function of P and ψ, i.e. $\chi = \chi(P,\psi)$, and write

$$\vec{\nabla}\chi = \frac{\partial\chi}{\partial P}\vec{\nabla} P + \frac{\partial\chi}{\partial\psi}\vec{\nabla}\psi \quad . \tag{17}$$

Substitution of this relation into Eq.(16) then yields

$$\frac{\partial\chi}{\partial P} = -\frac{1}{nq} \tag{18}$$

$$\frac{\partial\chi}{\partial\psi} = \frac{1}{c}\left(\frac{V_\phi}{2\pi r} - \frac{F\Omega_\phi}{rn}\right) \quad . \tag{19}$$

Since we are assuming the equation of state (13), Eq.(18) can be used to determine the modified electrostatic potential χ as a function of P and ψ; there is an integration constant which we determine by assuming that the density profile $n_0(\psi)$ is known in the absence of the plasma flow and the electrostatic potential, in which case χ vanishes by definition. The determination of χ for two specific cases, the isothermal case and the adiabatic case, is given in Appendix B.

Before using Eq.(19), we determine v_ϕ and Ω_ϕ by substituting Eqs.(11), (14) and (15) into the definition of $\vec{\Omega}$, i.e. Eq.(8), obtaining

$$\frac{1}{2\pi r}\frac{\partial\psi}{\partial z} = \frac{\partial v_\phi}{\partial z}\frac{mc}{q} + \frac{\partial A_\phi}{\partial z} \tag{20}$$

$$\Omega_\phi = B_\phi - \{\frac{\partial}{\partial z}\left[\frac{F(\psi)}{nr}\frac{\partial\psi}{\partial z}\right] + \frac{\partial}{\partial r}\left[\frac{F(\psi)}{nr}\frac{\partial\psi}{\partial r}\right]\}\frac{mc}{q} \tag{21}$$

$$\frac{1}{2\pi r}\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}(r\, v_\phi)\frac{mc}{q} + \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) \quad , \tag{22}$$

where we introduced the ϕ-component of the vector potential by the standard relations,

$$B_r = -\frac{\partial A_\phi}{\partial z} \quad , \qquad B_z = \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) \tag{23}$$

Equations (20) and (22) can readily be integrated to give

$$\psi = 2\pi r\left[\frac{mc}{q}v_\phi + A_\phi\right] + c_0 \quad , \tag{24}$$

where c_0 is a constant which we can choose to be zero without loss of generality. Equations (21) and (24) determine Ω_ϕ and v_ϕ in terms of the magnetic field and the known functions, $n(P,\psi)$ and $F(\psi)$.

Having expressed all fluid quantities, n, $\vec{v}$, χ and Ω_ϕ, in terms of the electromagnetic variables and the known functions of P and ψ, we can now use Eq.(19) and the definition of χ, i.e. Eq.(7), to determine P and ψ as functions of r, z, Ψ, A_ϕ and B_ϕ. These equations can be written as follows:

$$\frac{\partial \chi}{\partial \psi} = \frac{q}{4\pi^2 r^2 c^2 m} (\psi - \psi_0)$$

$$- \frac{Fm}{qrn} \{ \frac{q}{mc} B_\phi - \frac{\partial}{\partial z} [\frac{F}{nr} \frac{\partial \psi}{\partial z}] - \frac{\partial}{\partial r} [\frac{F}{nr} \frac{\partial \psi}{\partial r}] \} \tag{25}$$

$$\chi = \Psi + \frac{m}{2q} \{ (\frac{F}{rn})^2 [(\frac{\partial \psi}{\partial r})^2 + (\frac{\partial \psi}{\partial z})^2] + \frac{q^2 (\psi - \psi_0)^2}{4\pi^2 r^2 m^2 c^2} \} \tag{26}$$

where ψ_0 is the usual magnetic flux function,

$$\psi_0 = 2\pi r A_\phi \quad . \tag{27}$$

IV DIFFERENTIAL EQUATIONS FOR TWO-FLUID EQUILIBRIUM

The electromagnetic variables, Ψ , A_ϕ or ψ_0 and B_ϕ, which we assumed to be given in Sec.III, are to be determined self-consistently from the Maxwell and Poisson equations.

First, the electrostatic potential Ψ is determined from the Poisson equation (4), which in the cylindrical coordinate system reads,

$$\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r} \frac{\partial \Psi}{\partial r} = - 4\pi \sum_{j=i,e} q_j n_j \quad . \tag{28}$$

The magnetic field is determined from the Ampere equation (5). The r, z-components of this equation can easily be integrated to give

$$B_\phi = \frac{4\pi}{cr} [\sum_{j=e,i} q_j \int F_j(\psi_j) d\psi_j + I_0/2\pi] \tag{29}$$

where I_0 is an integration constant which corresponds to the externally applied axial current that can produce the toroidal magnetic field in the absence of the plasma. This equation determines the toroidal magnetic field in terms of the known functions. The ϕ-component of Eq.(5) can be written as

$$\frac{\partial^2}{\partial z^2}(rA_\phi) + \frac{\partial^2}{\partial r^2}(rA_\phi) - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)$$

$$= -\frac{2}{c^2}\sum_{j=e,i}\frac{q_j^2 n_j}{m_j}[\psi_j - 2\pi r A_\phi] \quad , \tag{30}$$

or using ψ_0 defined by (27) ,

$$\frac{\partial^2\psi_0}{\partial z^2} + \frac{\partial^2\psi_0}{\partial r^2} - \frac{1}{r}\frac{\partial\psi_0}{\partial r} = -\frac{4\pi}{c^2}\sum_{j=i,e}\frac{q_j{}^2 n_j}{m_j}(\psi_j - \psi_0) . \tag{31}$$

Combining Eq.(29) with Eq.(25), we can eliminate B_ϕ to obtain

$$n_j\frac{\partial\chi_j}{\partial\psi_j} - \frac{F_j}{q_j}\vec{\nabla}\cdot[\frac{m_j F_j}{n_j r^2}\vec{\nabla}\psi_j] - \frac{n_j q_j}{4\pi^2 r^2 c^2 m_j}(\psi_j - \psi_0)$$

$$= -\frac{4\pi}{r^2c^2}F_j[\sum_{\ell=i,e} q_\ell\int F_\ell(\psi_\ell)d\psi_\ell + I_0/2\pi] , \tag{32}$$

where we used the relation

$$\frac{1}{r}\frac{\partial}{\partial z}(\frac{F}{nr}\frac{\partial\psi}{\partial z}) + \frac{1}{r}\frac{\partial}{\partial r}(\frac{F}{rn}\frac{\partial\psi}{\partial r}) = \vec{\nabla}\cdot(\frac{F}{r^2 n}\vec{\nabla}\psi) .$$

The set of partial differential equations, (28), (31) and (32), now completely determine the two-fluid equilibrium of an axially symmetric plasma as expressed by the spatial distributions of ψ_i, ψ_e, ψ_0 and Ψ . In these equations, the functions $F_j(\psi_j)$ are assumed to be known, since they correspond to the poloidal drift fluxes which are to be determined by the plasma source and the diffusion process;

the density n_j is known as a function of P_j and ψ_j from the equation of state (13); the modified electrostatic potential χ_j is determined as a function of P_j and ψ_j by integration of Eq.(18) under the condition that the density distribution in the absence of Ψ and $\vec{v}_j$ is known; and the pressure P_j is determined as a function of ψ_j, ψ_0 and Ψ by the relation (26).

V. VARIATION PRINCIPLE

In this section, we show that the system of differential equations, (29), (31) amd (32), can be derived from a variation principle. Namely, we introduce the Lagrangean defined by

$$L[\psi_i, \psi_e, \psi_0, \Psi] \equiv \int_V r dr\, dz [\sum_{j=i,e} \{P_j + \frac{m_j F_j^2}{r^2 n_j} |\vec{\nabla}\psi_j|^2\}$$

$$+ \frac{1}{8\pi}\{|\vec{\nabla}\Psi|^2 + \frac{16\pi^2}{c^2 r^2}(\sum_{j=i,e} q_j \int F_j(\psi_j) d\psi_j + I_0/2\pi)^2$$

$$- \frac{1}{4\pi^2 r^2}|\vec{\nabla}\psi_0|^2\} \tag{33}$$

and show that

$$\delta L = 0 \tag{34}$$

yields the equations (29), (31) and (32). In the integrand of Eq.(33), the first line is the sum of the kinetic pressure P_j and the dynamical pressure due to the poloidal motion of the plasma, $n_j m_j (v_{jr}^2 + v_{jz}^2)$; the second line represents the electrostatic pressure, $|\vec{\nabla}\Psi|^2/8\pi$, and the toroidal magnetic pressure, $B_\phi^2/8\pi$ (see Eq.(29)); and the last line arises from the poloidal magnetic pressure, $(B_r^2 + B_z^2)/8\pi$. In this equation, P_j and n_j are determined from Eqs.(13), (18) and (26).

We first take the variation of Eq.(26); noting the relation

$$\delta\chi_j(p_j, \psi_j) = \frac{\partial\chi_j}{\partial P_j}\,\delta P_j + \frac{\partial\chi_j}{\partial\psi_j}\,\delta\psi_j$$

and using (18) and (26), we have

$$-\frac{1}{q_j n_j}\,\delta P_j + \frac{\partial\chi_j}{\partial\psi_j}\,\delta\psi_j$$

$$= \delta\Psi + \frac{m_j}{2q_j}\left\{\frac{2}{r^2 n_j}\,\delta\left(\frac{1}{n_j}\right)F_j^{\,2}(\vec\nabla\,\psi_j)^2 + \frac{1}{r^2 n_j^2}\,\delta(F_j^{\,2}|\vec\nabla\psi_j|^2)\right.$$

$$\left. + \frac{2q^2}{4\pi^2 r^2 m^2 c^2}(\psi_j - \psi_0)(\delta\psi_j - \delta\psi_0)\right\}. \qquad (35)$$

Therefore, the variation of the first line of Eq.(33) can be calculated as

$$\delta\left\{P_j + \frac{m_j F_j^{\,2}}{r^2 n_j}|\vec\nabla\psi_j|^2\right\} = q_j n_j \frac{\partial\chi_j}{\partial\psi_j}\,\delta\psi_j$$

$$- q_j n_j\,\delta\Psi + \frac{m_j}{2r^2 n_j}\,\delta\,(F_j^{\,2}\,|\vec\nabla\psi_j|^2)$$

$$- \frac{n_j q_j^{\,2}}{4\pi^2 r^2 m_j c^2}(\,\psi_j - \psi_0)(\delta\,\psi_j - \delta\psi_0\,). \qquad (36)$$

The third term on the right-hand side can alternatively be written as

$$\frac{m_j}{r^2 n_j}\cdot F_j(\vec\nabla\psi_j)\cdot\delta(\vec\nabla\int^{\psi_j} F_j(\psi_j)\,d\psi_j).$$

The variation of the remaining terms in Eq.(33) is straightforward. Setting $\delta\psi_j = \delta\,\psi_0 = \delta\Psi = 0$ on the surface of the plasma, we then obtain after integration by parts,

$$\delta L = \int_V rdrdz \Big[\sum_{j=i,e} \delta\psi_j \{ q_j n_j \frac{\partial \chi_j}{\partial \psi_j} - F_j \vec{\nabla} \cdot (\frac{m_j}{r_j{}^2 n_j} F_j \quad \vec{\nabla}\psi_j)$$

$$- \frac{n_j q_j^2}{4\pi^2 r^2 m_j c^2} (\psi_j - \psi_0) + \frac{4\pi}{c^2 r^2} (\sum_{\ell=i,e} q_\ell \int F_\ell d\psi_\ell + I_0/2\pi) q_j F_j \}$$

$$+ \delta\Psi \{ - \sum_{j=i,e} q_j n_j - \frac{1}{4\pi} \nabla^2 \Psi \} + \delta\psi_0 \{ \sum_{j=i,e} \frac{n_j \; q_j{}^2 (\psi_j - \psi_0)}{4\pi^2 r^2 m_j c^2}$$

$$+ \frac{1}{16\pi^3} \vec{\nabla} \cdot (\frac{1}{r^2} \vec{\nabla}\psi_0) \} \Big] \; . \qquad (37)$$

Equation (34) then yields the equations (32), (28) and (31).

We note that the variation principle given here is subject to no constraints other than the conditions that ψ_j, Ψ and ψ are fixed on the plasma surface. Therefore, this variation principle will be suited for numerical and perturbational analyses of the plasma equilibrium.

IV. CONCLUSIONS

We have derived a set of partial differential equations which determine the two-fluid equilibrium of an axially symmetric plasma. The equilibrium conditions are determined by the system of equations (28), (31) and (32) for ψ_i, ψ_e , ψ_0 and Ψ, supplemented by the relations (13), (18) and (26). The fluid and the electromagnetic variables, n_j, $\vec{v}_j$, P_j, $\vec{B}$ and $\vec{E}$, can be expressed in terms of ψ_i, ψ_e, ψ_0 and Ψ by (13), (14), (15), (18), (24), (25), (29) and the relation $\vec{E} = - \vec{\nabla}\Psi$. The system of differential equations (28), (31) and (32) can be formulated in the form of a variation principle (33) and (34). This variation principle is free from constraints and hence will be suitable for numerical and perturbational analyses.

The inertia force due to the plasma flow is taken into account by introducing the modified electrostatic potential and the modified magnetic field. The electrostatic potential is determined self-consistently by the Poisson equation and the equipotential surface is in general independent of the magnetic surface, in contrast to the case of the one-fluid MHD equilibrium. The method presented here can readily be generalized to a multi-component plasma, such as the D-T mixture.

The present method can be applied to various different situations. It can be used either for a toroidal plasma or for other types of axially symmetric plasmas. Application to a helically symmetric plasma as well as to an axially symmetric plasma with an oscillating field will be discussed in forthcoming papers.

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APPENDIX A

FORMULATION OF THE IDEAL MHD EQUILIBRIUM

The basic equations for the ideal MHD equilibrium are given by

$$\vec{\nabla}\cdot(\rho\vec{v}) = 0 \tag{A-1}$$

$$(\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}P + \frac{1}{4\pi\rho}(\vec{\nabla}\times\vec{B})\times\vec{B} \tag{A-2}$$

$$-\vec{\nabla}\Psi + \frac{1}{c}\vec{v}\times\vec{B} = 0 \tag{A-3}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{A-4}$$

where ρ is the plasma density. From the condition (A-4) and the axial symmetry, we can introduce the flux function ψ by the relation

$$\vec{B} = \left(-\frac{1}{2\pi r}\frac{\partial\psi}{\partial z}\ ,\quad B_\phi\ ,\quad \frac{1}{2\pi r}\frac{\partial\psi}{\partial r}\right)\ . \tag{A-5}$$

Using (A-1), (A-3) and (A-5), we find

$$\vec{v} = \left(-\frac{F(\psi)}{r\rho}\frac{\partial\psi}{\partial z}\ ,\quad v_\phi\ ,\quad \frac{F(\psi)}{r\rho}\frac{\partial\psi}{\partial r}\right) \tag{A-6}$$

$$\frac{v_\phi}{2\pi r} - \frac{F(\psi)B_\phi}{r\rho} = G(\psi) \tag{A-7}$$

$$\Psi = \frac{1}{c}\int G(\psi)\,d\psi\ , \tag{A-8}$$

where $F(\psi)$ and $G(\psi)$ are arbitrary functions of ψ which are undetermined within the framework of the ideal MHD theory.

Substituting (A-6) into (A-2), we get

$$\frac{1}{2}\vec{\nabla}|\vec{v}|^2 - \frac{v_\phi}{r}\vec{\nabla}(rv_\phi) - \frac{F}{\rho}\vec{\nabla}\left(\frac{F}{r^2\rho}\vec{\nabla}\psi\right)\vec{\nabla}\psi$$

$$= -\frac{1}{\rho}\vec{\nabla}P - \frac{1}{16\pi^3\rho}\vec{\nabla}\left(\frac{1}{r^2}\vec{\nabla}\psi\right) - \frac{B_\phi}{4\pi r\rho}\vec{\nabla}(rB_\phi) \tag{A-9}$$

$$\frac{F}{r^2\rho}\frac{\partial(\psi, rv_\phi)}{\partial(r,z)} = \frac{1}{8\pi^2 r^2\rho}\frac{\partial(\psi, rB_\phi)}{\partial(r,z)} . \tag{A-10}$$

From Eq. (A-10), we have

$$F r v_\phi - \frac{rB_\phi}{8\pi^2} = H(\psi) \tag{A-11}$$

where $H(\psi)$ is another arbitrary function of ψ. Substituting (A-7) and (A-11) into (A-9) yields

$$0 = \vec{\nabla}\left\{-\frac{1}{2}|\vec{v}|^2 + 2\pi G r v_\phi - \int \frac{dp}{\rho(p,\psi)}\right\}$$

$$+ \left[\frac{F}{\rho}\vec{\nabla}\left(\frac{F}{r^2\rho}\vec{\nabla}\psi\right) + \frac{\partial}{\partial\psi}\int \frac{dp}{\rho(p,\psi)}\right.$$

$$- \frac{1}{16^3\rho}\vec{\nabla}\left(\frac{F}{r^2}\vec{\nabla}\psi\right) - \frac{B}{4\pi r\rho}\left\{8\pi^2 r v_\phi \frac{dF}{d\psi} - 8\pi^2\frac{dH}{d\psi}\right\}$$

$$\left. - 2\pi r\, v_\phi \frac{dG}{d\psi}\right]\vec{\nabla}\psi \quad , \tag{A-12}$$

where we assumed that the plasma density ρ is a known function of the pressure P and the flux function ψ. We can write $\vec{\nabla}$ in the first line of Eq. (A-12) as $\vec{\nabla}\psi\, d/d\psi$. Then using (A-6), we finally obtain the following equation to determine the flux function,

$$\vec{\nabla}\left(\frac{1}{r^2}\vec{\nabla}\psi\right) - 16\pi^3 F\,\vec{\nabla}\left(\frac{F}{r^2\rho}\vec{\nabla}\psi\right)$$

$$= 16\pi^3\rho\left[\frac{\partial}{\partial\psi}\int\frac{dp}{\rho(p,\psi)} + 2\pi\frac{B_\phi}{r\rho}\left(\frac{dH}{d\psi} - r\, v_\phi\frac{dF}{d\psi}\right)\right.$$

$$\left. - 2\pi r\, v_\phi\frac{dG}{d\psi} - \frac{dR(\psi)}{d\psi}\right] \quad , \tag{A-13}$$

where

$$R(\psi) = \frac{1}{2}\left\{ \frac{F^2}{r^2\rho^2} |\vec{\nabla}\psi|^2 + v^2_\phi \right\} + \int \frac{dp}{\rho(p,\psi)} - 2\pi G r v_\phi \quad , \tag{A-14}$$

and B_ϕ, v_ϕ are, respectively, determined from (A-11) and (A-7) as

$$rB_\phi = \frac{8\pi^2[2\pi r^2 GF - H]}{1 - 16\pi^3F^2/\rho} \quad , \tag{A-15}$$

$$rv_\phi = \frac{2\pi[r^2G - 8\pi^2FH/\rho]}{1 - 16\pi^3F^2/\rho} \quad . \tag{A-16}$$

Note that $R(\psi)$ can contain an arbitrary constant which we can choose to be zero without loss of generality. Equation (A-14) can be regarded as the Bernoulli equation for the fluid.

It is easy to rewrite the differential equation (A-13) in the form of a variation principle. Indeed, by introducing the Lagrange function

$$L[\psi] = \int rdrdz\left[P + \frac{F}{r^2\rho^2}|\vec{\nabla}\psi|^2 + \frac{8\pi^3(2\pi r^2 GF-H)^2}{r^2(1-16\pi^3F^2/\rho)^2} - \frac{1}{32\pi^3r^2}|\nabla\psi|^2\right] \quad , \tag{A-17}$$

we can easily show that

$$\delta L[\psi] = 0 \tag{A-18}$$

yields the equation (A-13) with (A-15) and (A-16). The Lagrange function (A-17) can alternatively be written as

$$L[\psi] = \int rdr\, dz[P + \rho|\vec{v}_p|^2 + \frac{1}{8\pi}(B_\phi^{\,2} - |\vec{B}_p|^2)] \tag{A-19}$$

where $\vec{v}_p$ and $\vec{B}_p$ are the poloidal components of the velocity $\vec{v}$ and the magnetic field $\vec{B}$, respectively.

The similarity of this Lagrangean to that for the two-fluid is self-evident.

APPENDIX B

EXAMPLES OF THE MODIFIED ELECTROSTATIC POTENTIAL

The modified electrostatic potential $\chi(p,\psi)$ is determined from Eq.(18) and the equation of state. Here we derive the explicit expression for χ for two typical cases, the isothermal case and the adiabatic case.

We first consider the case where the temperature is uniform over the modified magnetic surface, i.e.

$$P = n\,T(\psi) \quad . \tag{B-1}$$

Substituting this relation into Eq.(18) gives

$$\chi(P,\psi) = -\frac{T(\psi)}{q}\,\ell n\,\frac{P}{n_0(\psi)T(\psi)} \quad , \tag{B-2}$$

where $n_0(\psi)$ is the density in the absence of the electrostatic potential and the plasma motion. Substituting (B-2) into (7), we can determine the equilibrium plasma density in the form

$$n=n_0(\psi)\exp[-\frac{q}{T(\psi)}\{\Psi + \frac{m}{2q}(\frac{F(\psi)^2}{r^2n^2}|\vec{\nabla}\psi|^2 + \frac{q^2(\psi-\psi_0)^2}{4\pi^2r^2mc^2})\}\,] . \tag{B-3}$$

This equation is nonlinear with respect to the density due to the presence of a poloidal flux $F(\psi)$. In general, there exist two solutions for n for given $n_0(\psi)$ and $F(\psi)$, provided that the following condition is satisfied,

$$\frac{n}{n_0}\exp[(q\Psi + \frac{m}{2}v_\phi^{\,2})/T] \leq e^{-1/2} \quad . \tag{B-4}$$

Conversely, if this condition is not satisfied, there is no equilibrium solution.

We next consider the case where the plasma motion is sufficiently rapid and the process becomes adiabatic. In this case we have[14]

$$\frac{3}{2}\, n\vec{v}\cdot\vec{\nabla}T + p\,\vec{\nabla}\cdot\vec{v} = 0 \quad . \tag{B-5}$$

Substituting the relation (14) and (15) as well as the relation T = P/n, we obtain

$$\frac{F}{r}\frac{\partial\psi}{\partial r}\left\{\frac{3}{2}\frac{\partial}{\partial z}\left(\frac{p}{n}\right)+p\frac{\partial}{\partial z}\left(\frac{1}{n}\right)\right\} - \frac{F}{r}\frac{\partial\psi}{\partial z}\left\{\frac{3}{2}\frac{\partial}{\partial r}\left(\frac{p}{n}\right)+p\frac{\partial}{\partial z}\left(\frac{1}{n}\right)\right\} = 0 \quad . \tag{B-6}$$

Use of Eq.(18) for n in (B-6) then yields

$$\frac{F}{r}\frac{\partial(\psi,\ p)}{\partial(r,z)}\left[\frac{5}{2}p\frac{\partial^2\chi}{\partial p^2} + \frac{3}{2}\frac{\partial\chi}{\partial p}\right] = 0 \quad ,$$

or noting (12) we have

$$\chi = \chi_0(\psi) - \chi_1(\psi)p^{2/5} \quad . \tag{B-7}$$

where $\chi_0(\psi)$ and $\chi_1(\psi)$ are arbitrary functions of ψ . The density n is determined by substitution of (B-7) into (18) as

$$n = \frac{5}{2}\,\frac{1}{q\ \chi_1(\psi)}\,p^{3/5} \quad . \tag{B-8}$$

Knowing $n_0(\psi)$ and $T_0(\psi)$ in the absence of Ψ and $\vec{v}$, we can determine $\chi_1(\psi)$ which then determines $\chi_0(\psi)$ from (B-7) since χ vanishes in the absence of Ψ and $\vec{v}$.