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Solitons in Plasma and Other Dispersive Media

—— Dawn of Nonlinear Physics ——

Yoshi H. Ichikawa and Miki Wadati*

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Further communication about this report is to be sent to the Research Information Center, Institute of Plasma Physics Nagoya University, Nagoya, Japan.

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* Department of Physics and Theoretical Physics Institute, University of Alberta, Edmonton, Alberta, Canada
On leave of absence from Institute for Optical Research, Kyoiku University, Tokyo

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Abstract

A review is given to recent development of extensive studies of nonlinear waves with purpose of showing methods of systematic analysis of nonlinear phenomena has been now established on the basis of new concept "soliton".

Firstly, characteristic properties of various kinds of solitons are discussed with illustration of typical nonlinear evolution equations. Brief discussions are also given to basic mechanisms which ensure the remarkable stability and individuality of solitons.

The reductive perturbation theory is a key method to reduce a given nonlinear system to a soliton system. Introductory survey is presented for an example of ionic mode in plasmas, although the method can be applied to any dispersive medium.

Central subject of the present review is the analytical methods of solving nonlinear evolution equations. The inverse method, the Bäcklund transformation and the concervation laws are discussed to emphasize that very firm analytical basis is now available to disentangle the nonlinear problems.

Finally, a notion of "dressed" solitons is introduced on basis of the higher order analysis of the reductive perturbation theory. In spite of the fact that success is restricted so far only for the one dimensional system, the achievement of soliton physics encourages us to face dawn of nonlinear physics with a confident expectation for forthcoming break through in the field.

Contents:

§1. Introduction

Modern physics has been so far developed on the basis of recognition of "linear" characteristics of the natural phenomena. Were confronted with a complicated dynamical system, we are accustomed to look for fundamental modes by decomposing the system into a part composed of freely moving components and an interaction among themselves. Putting faith in that this decomposition has physical meaning, we then investigate effects of the interaction among various modes as perturbation. In many fields of physics, however, we have been bewildered by the fact that this perturbation approach often meets serious difficulties such as divergent results of each order of perturbation and the lack of convergence of the perturbation series.

In nature, the success of a linearized theory is rather exceptional and in most cases the nonlinearity plays an essential role. Analysis of the dynamical behavior of nonlinear system is certainly one of the hardest problems. While most physicists admit the importance of the study of nonlinear system, the lack of a guiding principle, such as a superposition principle in the linear theory, have prevented us from developing a systematic research in the full variety of nonlinear phenomena. During the last decade, extensive studies on nonlinear waves in dispersive media such as plasmas have clarified remarkable properties of solitary wave. Thus, we expect hopefully that the key concept "soliton" may open the door for the establishment of nonlinear physics.

In the present paper, the status quo of the theory of nonlinear waves in dispersive media is summarized. Recently, the soliton turns out to be one of the most popular concept in theoretical and experimental physics and the study of soliton covers almost every field of physics. Therefore, it will be appropriate to start our discussion from the brief survey of history. Details of the concepts and methods will be found in the succeeding sections.

The soliton is very new concept, but we can trace back its seed to the old papers on hydrodynamics. According to a review paper by Scott, Chu and Mclaughlin[1), a story of the first discovery of solitary wave in a canal is described vividly by Scott-Russel. He observed a well-defined heap of water propagates along the canal without change of the form or diminution of the speed. It was an event of August, 1834. Pioneer works to give theoretical interpretation of the observation had been initiated by Boussinesq (1872) and by Korgeweg and de Vries (1895) at the end of the last century. In order to describe solitary waves in shallow water, they presented nonlinear partial differential equations which are now called by their names.

A real breakthough, however, came from the computer study of the Korteweg-de Vries equation carried out by Zabusky and Kruskal[2) in 1965. They found that the individual identity of steady solitary wave solution of the Korteweg-de Vries equation is preserved notwithstanding of their mutual interactions waves, and hence they proposed to call it "soliton".

They found also that an arbitrary shaped large amplitude wave is decomposed into a sequence of many solitons in the course of propagation. These discoveries immediately gave a clue to the famous Fermi-Pasta-Ulam problem in statistical mechanics. Fermi, Pasta and Ulam conjectured that the thermal equilibrium in solid is accomplished by the anharmonic interactions between atoms. Against their conjecture, they observed a recurrence phenomena in the one-dimensional nonlinear lattice. This recurrence phenomena now can be attributed to the particle-like properties of soliton. We will describe characteristic mechanisms to ensure the remarkable stability and individuality of various kinds of solitons in section 2.

Hope for development of systematic researches in nonlinear physics has been reinforced by the important fact that nonlinear evolution equations for the soliton systems are universal in many fields of physics. This has been recognized firstly by Gardner and Morikawa[3], in 1960, who showed that the hydromagnetic wave propagating with finite amplitude in perpendicular to the magnetic field is also described by the Korteweg-de Vries equation. Observing the close relationship between the linear dispersion relation and the structure of Gardner-Morikawa's transformation, Washimi and Taniuti[4] derived the Korteweg-de Vries equation for the ion-acoustic wave in cold plasma. The reductive perturbation theory developed by Taniuti and his collaborators[5] provides a systematic method to reduce relevant nonlinear evolution equations from the set of fundamental equations for a given physical system. In section 3, we present an instructive description of the reductive perturbation theory for the weakly dispersive system and for the strongly dispersive system. We will discuss the results of our recent analysis of the contribution of higher order terms of the reductive perturbation theory in section 5, where we introduce a notion of "dressed" soliton.

Finally, the dream of establishing nonlinear physics appears to be realized through vital efforts for search of analytic methods to solve the nonlinear evolution equations exactly. In 1971, Lamb rediscovered the Bäcklund transformation and obtained soliton solutions for the sine-Gordon equation[6]. A little earlier than Lamb's work, Gardner, Greene, Kruskal and Miura have developed a completely new method, called as the inverse scattering method, for solving the Korteweg-de Vries equation[7]. For nearly five years after their invention, the inverse scattering method has been believed to be effective only for the Korteweg-de Vries equation. Then, suddenly around 1972, people realized the inverse scattering method is not a fluke at all. Now we have more than 20 solvable soliton system. It should be emphasized that inverse scattering method has been a unique method whereby we can solve the initial value problem of nonlinear evolution equations. Furthermore, formalism of the method yields very naturally the proofs of the existence of an infinite number of conservation laws and complete integrability of the systems. We discuss in some details on these crucial aspects of the methods in section 4.

To conclude this introduction, we should remark also that the concept of soliton and new analytic methods are found to be applicable to discrete systems[8)] such as lattice system and particle system as well as continuous system. The interdisciplinary researches among physicists, mathematicians and engineers with vigorous curiosity have accelerated rapid development of the theory of soliton.

§2. Soliton

Let us start with giving a physical definition of soliton. Soliton is a nonlinear wave which possesses the following two properties: The first of them is

(1) A localized wave propagates without changing its properties such as the shape and the velocity, etc.

The typical example is a pulse-like wave shown in Fig.1. The celebrated Korteweg-de Vries equation;

Fig. 1

$$\frac{\partial}{\partial t} u - 6 u \frac{\partial}{\partial x} u + \frac{\partial^3}{\partial x^3} u = 0 \qquad (2.1)$$

has this shape of soliton. The Boussinesq equation

$$\frac{\partial^2}{\partial t^2} u - \frac{\partial^2}{\partial x^2} u - \frac{\partial^4}{\partial x^4} u - \frac{\partial^2}{\partial x^2} u^2 = 0 \quad . \qquad (2.2)$$

and the Modified Korteweg-de Vries equation

$$\frac{\partial}{\partial t} u + 6u^2 \frac{\partial}{\partial x} u + \frac{\partial^3}{\partial x^3} u = 0 \qquad (2.3)$$

also bear this type of soliton. Into the category of localized waves, we may include a wave of which shape is given by integrating a pulse-like wave as shown in Fig.2;

Fig. 2

$$\phi(x,\ t) = \int^{x} u(y,t)\,dy \qquad (2.4)$$

Sometimes, we call this shape of soliton "kink". For instance, the sine-Gordon equation;

$$\frac{\partial^2}{\partial t^2}\phi - \frac{\partial^2}{\partial x^2}\phi + \sin\phi = 0 \quad , \qquad (2.5)$$

or an equivalent equation expressed in a characteristic frame:

$$\frac{\partial^2}{\partial t \partial x}\phi + \sin\phi = 0 \qquad (2.6)$$

has this shape of soliton.

In the above examples, the wave fields $u(x,t)$ and $\phi(x,t)$ are real. In the case of complex field, envelop of the wave might be localized. We call this localized envelop of wave "envelop soliton". An example is the nonlinear Schrödinger equation;

$$i\frac{\partial}{\partial t}\psi + p\frac{\partial^2}{\partial x^2}\psi + q|\psi|^2\psi = 0 \ . \qquad (2.7)$$

In particular, a soliton for $pq > 0$ is called "bright" soliton (Fig.3(a)) and a soliton for $pq < 0$ is called "dark" soliton (Fig.3(b)). The words "bright" and "dark" come from an application of the nonlinear Schrödinger equation to nonlinear optics where $|\psi|^2$ stands for the intensity of electric field.[9] Fig.3(a) Fig.3(b)

The second property to be hold by soliton is

(2) Localized waves propagate without losing their individuality and they are stable in the processes of mutual collisions.

That is to say, soliton behaves as if it is a particle. In Fig.4, we illustrate an example of collision of two solitons. Each soliton conserves its identity before and after the collision although solitons interact strongly during collision. For the soliton systems described by eqs.(2.1) ~ (2.7), we can prove that collisions among N solitons are described as superpositions of successive binary collitions between solitons.

Fig. 4

Mathematically, we can give a rigorous definition of soliton system as a completely integrable system. In the language of statistical mechanics, the soliton system is non-ergodic. Corresponding to the situation that wave fields have infinite degree of freedom, the soliton system has such a remarkable property that an infinite number of conservation laws holds for the soliton system. Recently, we have shown that these conservation laws are closely related to the symmetries of the system.[10)]

Then, we turn to discuss what mechanisms are effective to sustain the shape of solitons and to insure their stability:

(a) The competitions of nonlinearity and dispersion sustain the stable wave forms.

Most of the solitons belong to this category. For instance, in the Korteweg-de Vries equation (eq.(2.1)), the second term is a nonlinear term and the third term represents dispersion effect. Nonlinear effect acts to steepen the wave, while dispersion effect make the wave to spread. Balance of these opposing effects secures the stable wave forms.

(b) Combined effect of nonlinear terms alone may keep stable wave forms.

An example is so-called three wave interaction process described by the following equations,[11)]

$$\frac{\partial}{\partial t}\psi_0 + v_{g_0}\frac{\partial}{\partial x}\psi_0 = -\beta_0\,\psi_1\,\psi_2^{*} \quad , \qquad (2.8a)$$

$$\frac{\partial}{\partial t}\psi_1 + v_{g_1}\frac{\partial}{\partial x}\psi_1 = -\beta_1\,\psi_0\,\psi_2 \quad , \qquad (2.8b)$$

$$\frac{\partial}{\partial t}\psi_2 + v_{g_2}\frac{\partial}{\partial x}\psi_2 = -\beta_2\,\psi_0^{*}\,\psi_1 \quad . \qquad (2.8c)$$

Here $\psi_j (j=0, 1,2)$ are complex amplitude of waves, which satisfy the resonance conditions $\omega_1 = \omega_0 + \omega_2$ and $k_1 = k_0 + k_2$. $v_{gj} (j=0, 1, 2)$ are group velocities and $\beta_j (j=0, 1, 2)$ are coupling constants with positive sign. In this case, a variation due to translation of a wave balances with a variation due to interaction between two other waves. This set of equations can be regarded to be an extension of Volterra equation which describes population competitions among species. Direct analogy holds between the process described by the set of equations (2.8a ~ c) and the process of self-induced transparency studied in the nonlinear optics.[12)] We illustrate the collision process between two triple solitary pulses in Fig.5, in which the amplitudes ψ_i are taken to be real.[13)] Fig. 5

(c) The third mechanism is a topological stability. This is now getting popularity in physics of elementary particles.[14)] As a mechanical analogue of the topological stability, we may present the sine-Gordon equation which describes wave motions in a series of pendulums connected by linear springs (Fig.6). One twist of string (say, clockwise twist) corresponds to a kink. Fig. 6

It is to be noticed certainly that the above classification is not strict and we have no intention to exclude other possibilities.

§3. Reductive Perturbation Theory

Nonlinear behaviours of dynamical systems are modeled by such nonlinear evolution equations as listed in the previous section. Various approaches have been proposed to reduce the nonlinear evolution equations from a set of fundamental equations describing dynamical behaviour of the given physical systems. Here, we present the reductive perturbation theory developed by Taniuti and his collaborators during the years of 1968~1974. Development of the theory has been inspired by the work of Gardner and Morikawa. Extending the concept of the far field of the wave equation, however, Taniuti and his collaborators have formulated the reductive perturbation theory as a method to reduce a general hyperbolic system to a single solvable nonlinear equation describing a far field of the system.

For the purpose of introduction to the reductive perturbation theory, we will recapitulate the approach of Gardner and Morikawa by applying their method to derivation of the Korteweg-de Vries equation for ion-acoustic wave in plasma. For a collisionless plasma composed by cold ions and warm electrons, the basic set of equations may be expressed as (in a dimensionless form);

$$\frac{\partial}{\partial t} n + \frac{\partial}{\partial x} (nu) = 0 \quad , \tag{3.1a}$$

$$\frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u = - \frac{\partial}{\partial x} \psi \ , \tag{3.1b}$$

$$\frac{\partial^2}{\partial x^2}\psi = n_e - n \quad , \tag{3.1c}$$

$$n_e = \exp(\psi) \quad , \tag{3.1d}$$

where $n = \tilde{n}_i/n_0$, $n_e = \tilde{n}_e/n_0$, $u = \tilde{u}(\kappa T_e/M)^{-1/2}$ and $\psi = \tilde{\phi}(\kappa T_e/e)^{-1}$ are the dimensionless ion number density, electron number density, ion velocity and electro-static potential, respectively. Dimensionless space-time variables (x,t) are measured by the Debye distance $(\kappa T_e/4\pi e^2 n_0)^{1/2}$ and the ion plasma frequency $(4\pi e^2 n_0/M)^{1/2}$. We impose the boundary condition;

$$n \to 1 \ , \qquad \psi \to 0, \qquad u \to u_0 \qquad \text{as } |x| \to \infty \quad . \tag{3.2}$$

Gardner and Morikawa looked for an approximate time dependent behaviour of nonlinear waves at large t under the conditions that (a) it should include the nonlinear stationary solution and (b) it should exhibit the similar asymptotic evolution as the linearized field does.

Thus, firstly let us look for the stationary solution of the set of equations (3.1a) ~ (3.1d). Under the boundary condition (3.2), it is straight forward to obtain

$$\frac{\partial^2}{\partial x^2}\psi = \exp(\psi) - (1 - \frac{2}{u_0{}^2}\psi)^{-1/2} \quad . \tag{3.3}$$

We consider ψ is small but finite (say, $\psi \sim O(\varepsilon)$, ε is a small parameter) and set

$$\xi = \varepsilon^{1/2} x \quad , \tag{3.4a}$$

$$u = 1 + \varepsilon a \quad . \tag{3.4b}$$

Up to the order of ε^2, eq.(3.3) yields

$$\frac{\partial^2}{\partial \xi^2} \Phi = - \Phi^2 + 2 a \Phi , \tag{3.5}$$

where

$$\psi = \varepsilon \Phi . \tag{3.6}$$

Under the boundary condition (3.2), eq.(3.5) is integrated to give

$$\psi = \psi_0 \operatorname{sech}^2 \left(\sqrt{\frac{\Phi_0}{6}} \xi \right) , \tag{3.7}$$

representing a localized solitary wave.

Secondaly, we turn to examine asymptotic behavior of the linearized field at large t. Linearization of the system (3.1a ~ 1d) yields

$$\frac{\partial^2}{\partial t^2} \psi - \frac{\partial^2}{\partial x^2} \psi - \frac{\partial^4}{\partial t^2 \partial x^2} \psi = 0 . \tag{3.8}$$

The solution of eq.(3.8) with the initial condition

$$\psi(x, t = 0) = \frac{\partial}{\partial t} \psi(x, t = 0) = 0 \tag{3.9}$$

and the boundary condition

$$\psi(0, t) \begin{cases} = \psi_0 , & t \geqslant 0 \\ = 0 & t < 0 \end{cases} \tag{3.10}$$

is obtained as

$$\psi = \psi_0 \frac{1}{2\pi i} \int ds \quad s^{-1} \exp\{ts[1-\alpha(1 + s^2)^{-1/2}]\} , \tag{3.11}$$

of which asymptotic evolution at large t is given as

$$\psi \sim \int_{\beta}^{\infty} Ai(\alpha) d\alpha , \tag{3.12}$$

where

$$\alpha = \frac{x}{t} \quad , \tag{3.13a}$$

$$\beta = \frac{2}{3}\,(x - t)/t^{1/3} \quad , \tag{3.13b}$$

$$\mathrm{Ai}(\alpha) = \frac{1}{\pi}\int_0^\infty \cos(\frac{1}{3}\,\lambda^3 + \alpha\lambda\,)\,d\lambda \quad . \tag{3.13c}$$

Gardner and Morikawa suggest us to look for an approximate equation which describes evolution of the linear asymptotic solution (3.12) into the nonlinear stationary solution (3.7).

Upon observing at the expressions of eqs.(3.4a) and (3.13b), we notice that the condition (a) and (b) are satisfied if we set

$$\xi = \varepsilon^{1/2}(x - t) \quad , \tag{3.14a}$$

$$\tau = \varepsilon^{3/2}\,t \quad . \tag{3.14b}$$

With these re-scaling of the independent variables, the basic equations (3.1a ~ d) are transformed as follows.

$$\varepsilon\frac{\partial}{\partial\tau}\,n - \frac{\partial}{\partial\xi}\,n + \frac{\partial}{\partial\xi}\,(nu) = 0 \quad , \tag{3.15a}$$

$$\varepsilon\frac{\partial}{\partial\tau}\,u - \frac{\partial}{\partial\xi}\,u + u\,\frac{\partial}{\partial\xi}\,u = -\,\frac{\partial}{\partial\xi}\,\psi \quad , \tag{3.15b}$$

$$\varepsilon\frac{\partial^2}{\partial\xi^2}\,\psi = \exp(\psi) - n \quad . \tag{3.15c}$$

Substituting power series expansions of n, u and ψ ,

$$n = 1 + \varepsilon\, n^{(1)} + \varepsilon^2\, n^{(2)} + \cdots \tag{3.16a}$$

$$u = \varepsilon\, u^{(1)} + \varepsilon^2\, u^{(2)} + \cdots \tag{3.16b}$$

$$\psi = \varepsilon\psi^{(1)} + \varepsilon^2\psi^{(2)} + \cdots \tag{3.16c}$$

into eqs.(3.15a ~ c), we can establish relationships among the first order quantities as

$$\psi^{(1)} = u^{(1)} = n^{(1)} = n_e^{(1)} \tag{3.17}$$

at the level of the first order expansions of eqs.(3.15a ~ c). Their explicit (ξ,τ) dependence, however, is not determined at this level. Going up to the second order expansions of eqs.(3.15 ~ c), we obtain the Korteweg-de Vries equation for the first order potential

$$\frac{\partial}{\partial\tau}\psi^{(1)} + \frac{1}{2}\frac{\partial^3}{\partial\xi^3}\psi^{(1)} + \psi^{(1)}\frac{\partial}{\partial\xi}\psi^{(1)} = 0 \tag{3.18}$$

as the compativility condition among the second order components of eqs.(3.15a ~ c).

Taniuti and Wei[15] have established the reductive perturbation method for the nonlinear propagation of long waves by applying the Gardner-Morikawa transformation,

$$\xi = \varepsilon^a(x - \lambda_0 t) \quad , \tag{3.19a}$$

$$\tau = \varepsilon^{1+a}\, t \quad , \tag{3.19b}$$

$$a = \frac{1}{p-1} \quad , \tag{3.19c}$$

to a set of equations for a vector U with n-components,

$$\frac{\partial}{\partial t}U+A(U)\frac{\partial}{\partial x}U + \{\sum_{\beta=1}^{s}\prod_{\alpha=1}^{p}(H_\alpha^\beta(U)\frac{\partial}{\partial t} + K_\alpha^\beta(U)\frac{\partial}{\partial x})\}\cdot U=0,(p\geqslant 2) \tag{3.20}$$

where A, H_α^β and K_α^β are $n \times n$ matrices, being functions of U. Expanding these quantities around a constant solution U_0 in a small parameter ε, they have reduced eq.(3.20) to

$$\frac{\partial}{\partial t} \psi^{(1)} + \mu \frac{\partial^p}{\partial \xi^p} \psi^{(1)} + \alpha \psi^{(1)} \frac{\partial}{\partial \xi} \psi^{(1)} = 0 \quad , \tag{3.21}$$

where $\psi^{(1)}$ is defined by

$$U^{(1)} = \psi^{(1)} R \tag{3.22}$$

with R the right eigenvector of A_0 , i.e.,

$$(A_0 - \lambda_0)R = 0 \quad . \tag{3.23}$$

The coefficients μ and α are given by the relations

$$\mu = (L\, K_0\, R)/(L \cdot R) \quad , \tag{3.24a}$$

$$\alpha = L(R \cdot \nabla_u A_0)R/(L \cdot R), \tag{3.24b}$$

where L stands for the left eigen vector of A_0 and

$$K_0 = \Sigma\, \Pi\, (-\lambda_0 H^{\alpha}_{\beta_0} + K^{\alpha}_{\beta_0}) \quad . \tag{3.25}$$

It is straightforward to see λ_0 and μ are the phase velocity and the dispersion (or dissipation) coefficient of the linear dispersion relation

$$\omega = \lambda_0 k + \mu\, i^{p-1} k^p + 0(k^{2p-1}), \tag{3.26}$$

reduced from eq.(3.20).

The crucial assumption in the above reduction is that there exists at least one real and non-degenerate eigenvalue of A_0, which is denoted by λ_0 . Furthermore, in the practical application of the method, we encounter exceptional cases in which either

$$\alpha = 0 \quad , \quad \text{i.e.,} \quad \nabla_u \lambda_0 \cdot R = 0 \quad , \tag{3.27a}$$

or

$$\mu = 0 \quad , \quad \text{i.e.,} \quad L \cdot K_0 \cdot R = 0 \quad , \tag{3.27b}$$

The former case occurs for the Alfvén wave, and the latter is a case of the magneto-acoustic wave. The necessary modifications for these exceptional cases have been accomplished by Kakutani and his collaborators.[16)]

Now, turning to our interest to the case of strongly dispersive waves, we describe briefly a generalization of the method to a wide class of nonlinear wave systems. Taniuti and Yajima[17)] considered a system of equations,

$$\frac{\partial}{\partial t} U + A(U) \frac{\partial}{\partial x} U + B(U) = 0 \tag{3.28}$$

Here, U is again a column vector with n components $u_1, u_2, \cdots u_n$, and A an $n \times n$ matrix and B a column vector. We assume eq.(3.28) has a constant state solution U_0, which satisfies

$$B(U_0) = 0 \quad . \tag{3.29}$$

Then, eq.(3.28) linearized about U_0 takes the form

$$\frac{\partial}{\partial t} U^{(1)} + A(U_0) \frac{\partial}{\partial x} U^{(1)} + \nabla_u B(U)_0 \cdot U^{(1)} = 0 \quad , \tag{3.30}$$

which admits a plane wave solution $U^{(1)} \sim \exp[i(kx - \omega t)]$ subject to the dispersion relation;

$$\det(\pm i(\omega I - k\, A(U_0)) + \nabla_u B(U)_0) = 0 \quad , \tag{3.31}$$

where I is the unit matrix, the (i, j) element of $\nabla_u B(U)_0$ is given as $(\partial B_i/\partial u_j)_{U=U_0}=0$. In terms of a small parameter ε, we introduce slow variables

$$\xi = \varepsilon(x - \lambda t) \quad , \tag{3.32a}$$

$$\tau = \varepsilon^2 t \quad . \tag{3.32b}$$

and assume U can be expanded about U_0 as follows;

$$U = U_0 + \sum_{\ell=-\infty}^{\infty} \sum_{\alpha=1}^{\infty} \varepsilon^{\alpha} U_{\ell}^{(\alpha)}(\xi,\tau)\exp[i\ell(kx-\omega t)] \ . \tag{3.33}$$

Substituting eqs.(3.32a ~ b) and (3.33) into the original equation (3.28), we obtain a set of equations corresponding to the each order of powers of ε and the ℓ-th harmonic component. In the first order of ε, the dispersion relation (3.31) assures that $U_{\pm 1}^{(1)} = \psi^{(1)} R$ with the right eigenvector R given by

$$[\mp i(\omega I - k\, A(U_0)) + \nabla_u B(U)_0]R = 0 \quad , \tag{3.34}$$

and $U_{\ell}^{(1)} = 0$ for $|\ell| \geqslant 2$. In the second order of ε, the $\ell = 1$ component yields a condition

$$\lambda \equiv \frac{\partial \omega(k)}{\partial k} \tag{3.35}$$

to deal with nontrivial case $\partial U_{\pm 1}^{(1)}/\partial\xi \neq 0$. The $\ell = 2$ component and the $\ell = 0$ component of the second order equation determines the second order beat wave $U_2^{(2)}$ and $U_0^{(2)}$. Finally, at the third order of ε, the $\ell = 1$ component gives rise to the nonlinear Schrödinger equation

$$i\frac{\partial}{\partial\tau}\psi^{(1)} + p\frac{\partial^2}{\partial\xi^2}\psi^{(1)} + q\,|\psi^{(1)}|^2\,\psi^{(1)} = 0 \quad . \tag{3.36}$$

It is not so difficult to show $p = (1/2)\partial^2\omega/\partial k^2$. Again the second term describes dispersion effect and the third term represents nonlinear effects.

In discussing nonlinear wave phenomena in collisionless plasma, it is very crucial to take into account contributions of the particle-wave resonant interactions. Sugihara and Taniuti[18] have examined structure of the resonant far fields of the Vlasov equation. Ichikawa and his collaborators[19] have examined contributions of the particle-wave-wave resonant interaction, while effects of the resonant particles to the ion-acoustic solitary waves has been analyzed in some details by Taniuti.[20]

§4. Analytical Methods of Solving Nonlinear Evolution Equations

In the previous section, we have discussed a systematic method to derive a nonlinear evolution equation from a basic set of equations for a given physical system. Concerned with thus derived nonlinear evolution equations, recent investigations have clarified that there are a number of nonlinear evolution equations which share the following common properties;

1) The equations can be solved exactly by the "inverse scattering method" and yield N-soliton solutions.
2) The equations have the "Bäcklund transformations" which transform equations to themselves.
3) The equations have an infinite number of "conservation laws".

We shall explain each statement and show these properties are closely related.

4.1). The inverse scattering method

Fundamental principle of the inverse scattering method may be summarized as follows.

1) Introduce an eigenvalue problem whose potential is the solution U(x,t) of a nonlinear evolution equation, which we want to solve. When we choose an operator of the eigenvalue problem properly, the eigenvalue remains to be time-invariant while u(x,t) evolves according to the nonlinear evolution equation.

2) Solving the inverse scattering problem (namely, determining a potential from scattering data) for the eigenvalue problem introduced above, we can reconstruct a sought function u(x,t) from the time-dependent scattering data.

Initial value problem of a nonlinear evolution equation

$$\frac{\partial}{\partial t} u = K[u] \tag{4.1}$$

is solved by the inverse scattering method as follows;

a) determine scattering data at t=0 from the initial value u(x,0),

b) evaluate a time dependence of the scattering data and determine scattering data at t=t,

c) reconstruct a potential u(x,t) from the scattering data at t=t.

If we regard scattering data as Fourier components, the above procedure may be interpreted as an extention of Fourier transformation to the nonlinear case.

To be specific, let us consider the Korteweg-de Vries equation;

$$\frac{\partial}{\partial t} u - 6\, u \frac{\partial}{\partial x} u + \frac{\partial^3}{\partial x^3} u = 0 \quad . \tag{4.2}$$

Associated with eq.(4.2), introduce the linear eigenvalue problem

$$L\ \psi = \lambda\ \psi \quad , \tag{4.3a}$$

with

$$L = -\frac{\partial^2}{\partial x^2} + u(x,\ t) \quad , \qquad -\infty < x < +\infty \quad . \tag{4.3b}$$

We impose the boundary condition, $u(x,t) \to 0$ as $|x| \to \infty$. Some lengthy calculation yields

$$\frac{\partial}{\partial t}\lambda \cdot \psi^2 + \frac{\partial}{\partial x}\left(\psi \frac{\partial}{\partial x} Q - \frac{\partial}{\partial x}\psi \cdot Q\right) = 0 \quad , \tag{4.4}$$

where

$$Q \equiv \frac{\partial}{\partial t}\psi + \frac{\partial^3}{\partial x^3}\psi - 3(u+\lambda)\frac{\partial}{\partial x}\psi \ . \tag{4.5}$$

If ψ vanishes sufficiently fast as $|x| \to \infty$, the second term of eq.(4.4) vanishes on integration over the interval $(-\infty,\infty)$. Thus, we can prove the important result $\partial\lambda/\partial t = 0$. Hence, we can integrate eq.(4.4) twice to yield

$$\frac{\partial}{\partial t}\psi + \frac{\partial^3}{\partial x^3}\psi - 3(u+\lambda)\frac{\partial}{\partial x}\psi = C(t) + D(t)\psi\int^{x}\psi^{-2}dx \ , \tag{4.6}$$

where C and D are integration constants.

Now, we shall find out the asymptotic behavior of $\psi(x,t)$ in regions where $u(x,t)$ vanishes, that is, at $|x| \to \infty$. For a discrete eigenvalue $\lambda_n = -\kappa_n^2$ (κ_n is real, $n=1,\ 2,\ \cdots N$), it is easy to show $C(t)=D(t)=0$ because of the boundary condition and the normalization condition of ψ_n. Then, eq.(4.6) gives

$$\psi_n(x,t) = C_n(t)\exp(\kappa_n x) \ , \qquad x \to -\infty \ . \tag{4.7}$$

where

$$c_n(t) = c_n(0)\exp(-4\kappa_n{}^3 t) \quad . \tag{4.8}$$

For the continuous eigenvalue $\lambda = k^2$ (k is real), the asymptotic form of $\psi(x, t)$ is given as follows when, in the case that the incident wave comes from the left,

$$\begin{aligned} \psi(x, t) &= a(k, t)\exp(i\ k\ x) \quad , \qquad x \to \infty \\ &= \exp(i\ k\ x) + b(k, t)\exp(-i\ k\ x), \quad x \to -\infty, \end{aligned} \tag{4.9}$$

where $a(k,t)$ and $b(k,t)$ are the transmission and reflection coefficient, respectively. Substitution of eq.(4.9) into eq.(4.6) yields $C(t)=-4ik^3$, $D(t)=0$, and

$$a(k, t) = a(k, 0) \tag{4.10a}$$

$$b(k, t) = b(k, 0)\exp(-8\ i\ k^3 t) \quad . \tag{4.10b}$$

We call the aggregate of quantities $\{\kappa_n, c_n, n=1, 2, \cdots N;\ a(k), b(k), k \text{ real}\}$ the scattering data. The potential $u(x,t)$ can be reconstructed from the scattering data at $t=t$ (the inverse scattering problem !) as

$$u(x, t) = 2\frac{d}{dx} K(x, x) \quad , \tag{4.11}$$

where the function $K(x,x)$ is a solution of the Gelfand-Levitan equation

$$R(x+y)+K(x,y)+\int_{-\infty}^{x} R(y+z)K(x,z)dz=0, \quad x \geq y, \tag{4.12}$$

with

$$\begin{aligned} R(x+y) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} b(k,t)\exp(-i\ k\ (x+y))dk \\ &+ \sum_{n=1} c_n{}^2(t)\exp(\kappa_n(x+y)) \quad . \end{aligned} \tag{4.13}$$

The Gelfand-Levitan equation is a linear integral equation, and can be solved easily to yield N-soliton solution in the reflectionless case $b(k) = 0$.[21]

In 1972, Zakharov and Shabat[22] showed that the nonlinear Schrödinger equation (2.7) can be solved by the inverse scattering method with introduction of the eigenvalue problem of 2×2 Dirac operator L. Subsequently, the modified Korteweg-de Vries equation (2.3) was shown to be solvable in a similar fashion.[23]

Then, one might ask, "For a given nonlinear evolution equation, what kind of the eigenvalue problem should be associated ? ". For the moment, we do not have any definite answer for this question. Conversely, Ablowitz, Kaup, Newell and Segur have examined for a given eigenvalue problem what kind of nonlinear evolution equations will be included. Following their approach[24], we consider the following eigenvalue problem

$$\frac{\partial}{\partial x} \psi_1 - \eta \psi_1 = q(x, t)\psi_2 , \tag{4.14a}$$

$$\frac{\partial}{\partial x} \psi_2 + \eta \psi_2 = r(x, t)\psi_1 . \tag{4.14b}$$

and specify time dependence of the eigenfunction ψ_1 and ψ_2 as follows;

$$\frac{\partial}{\partial t} \psi_1 = A(x,t,\eta)\psi_1 + B(x, t, \eta)\psi_2 , \tag{4.15a}$$

$$\frac{\partial}{\partial t} \psi_2 = C(x,t,\eta)\psi_1 - A(x, t, \eta)\psi_2 . \tag{4.15b}$$

The requirement of time independence of the eigenvalue η yields

$$\frac{\partial}{\partial x} A = qc - rB \quad , \tag{4.16a}$$

$$\frac{\partial}{\partial x} B - 27B = \frac{\partial}{\partial t} q - 2Aq \quad , \tag{4.16b}$$

$$\frac{\partial}{\partial x} C + 27C = \frac{\partial}{\partial t} r + 2Ar. \tag{4.16c}$$

The set of equations (4.16a ~ c) gives rise to various nonlinear evolution equations for proper choice of A, B and C. According to this formalism, the nonlinear evolution equations listed in the second section are classfied in the following manner;

a) r=constant; the Korteweg-de Vries equation, (2.1)

b) $r=-q$; the modified Korteweg-de Vries equation, (2.3),
the sine-Gordon equation (2.6),

c) $r=-q^*$; the nonlinear Schrödinger equation, (2.7).

More recently, the inverse scattering method are extended to cover wider classes of nonlinear evolution equations, such as the Boussinesq equation, (2.2), the three wave interaction equation, (2.8) etc. Also, it has been shown that the inverse scattering method works even in the case of periodic boundary condition.[25)]

4.2). The Bäcklund transformation

The Bäcklund transformation originates from the transformation theory in the differential geometry. In the theory of partitial differential equation, the Bäcklund transformation may be defined as follows. A Bäcklund transformation for a partial differential equation of the second order in two independent variables is a pair of the first order partial differential equations that relate the dependnet variable satisfying the given equation to another dependent variable which satisfies the same (or in general, another) partial differential equation of the second order.

Again, let us take the Korteweg-de Vries equation as an illustrative example. This time, we introduce a potential function $w(x,t)$, given by $u=w_x$, where the suffix x stands for the partial differentiation $\partial/\partial x$. Similarly we denotes the partial differentiation $\partial/\partial t$ by suffix t in the following. The Korteweg-de Vries equation is expressed as

$$w_{xt} - 6\, w_x\, w_{xx} + w_{xxxx} = 0 \quad . \tag{4.17}$$

The Bäcklund transformation for the Korteweg-de Vries equation is[26)]

$$w_x + w'_x = -2\eta^2 + \frac{1}{2}\,(w - w')^2 \quad , \tag{4.18a}$$

$$w_t + w'_t = 2(w_x^2 + w_x\, w'_x + w'^2_x) - (w-w')(w_{xx} - w'_{xx}) \quad , \tag{4.18b}$$

with the arbitrary constant η. In fact, it is readily shown that if w is a solution, w' also satisfies the Korteweg-de Vries equation. At first, we shall show that one-soliton solution comes very naturally from the transformation. The trivial solution of eq.(4.17) is w=u=0 (the "vacuum solution"). Substituting the vacuum solution into eq.(4.18) a ~ b), we obtain two types of solutions, regular and irregular one,

$$w' = -2\eta \tan h\,(\eta\xi), \quad u' = 2\,\eta^2 \mathrm{sech}^2(\eta\xi) \quad , \tag{4.19}$$

$$w'^* = -2\eta \coth(\eta\xi), \quad u'^* = 2\eta^2 \mathrm{cosech}^2(\eta\xi) \quad , \tag{4.20}$$

where $\xi = x - 4\eta^2 t + \xi_0$. The regular solution (4.19) and irregular solution (4.20) are called one-soliton solution and one-antisoliton solution, respectively. This process can be continued step by step.

More useful and important result from the Bäcklund transformation is an algebraic recursion formula for constructing a ladder of solutions. Let w_0, w_1, w_2 and w_{12} be the solutions of the Korteweg-de Vries equation connected by the Bäcklund transformation as follows

$$w_{0,x} + w_{1,x} = -2\eta_1^2 + \frac{1}{2}(w_0 - w_1)^2 , \tag{4.21a}$$

$$w_{0,x} + w_{2,x} = -2\eta_2^2 + \frac{1}{2}(w_0 - w_1)^2 , \tag{4.21b}$$

$$w_{1,x} + w_{12,x} = -2\eta_2^2 + \frac{1}{2}(w_1 - w_{12})^2 , \tag{4.21c}$$

$$w_{2,x} + w_{12,x} = -2\eta_1^2 + \frac{1}{2}(w_2 - w_{12})^2 . \tag{4.21d}$$

Eliminating the terms with x-derivative in eqs.(4.21a ~ d), we obtain

$$w_{12} = w_0 - 4\,\frac{\eta_1^2 - \eta_2^2}{w_1 - w_2} . \tag{4.22}$$

That is to say, the second-order solution w_{12} is expressed in terms of the original solution w_0 and the two first-order solution w_1 and w_2. Since the starting solution is not specified, this relation gives a recursion formula for constructing a ladder of solutions. For instance, if we take w_0 the vacuum solution, w_1 and w_2 one-soliton and one-antisoliton solution, respectively, w_{12} becomes two soliton solution. Eq.(4.22) is an example of nonlinear superposition principle.

A striking property of the Bäcklund transformation is its close relation to the inverse scattering method.[26),27)] Consider eqs.(4.18a ~ b). We introduce a function defined by

$$w' - w = -2 \quad \psi_x/\psi \quad . \tag{4.22}$$

Substitution of eq.(4.22) into eqs.(4.18a ~ b) gives rise to

$$-\psi_{xx} + u\,\psi = \lambda\,\psi\ , \quad \lambda = -\eta^2\ , \tag{4.23a}$$

$$\begin{aligned}\psi_t &= -u_x\psi + (2u + 4\lambda)\,\psi_x + C\,\psi \\ &= -\psi_{xxx} + 3(u + \lambda)\,\psi_x + C\,\psi\ .\end{aligned} \tag{4.23b}$$

These are nothing but the fundamental equations of the inverse scattering method, eqs.(4.3a ~ b) and (4.6).

Let us study one more example. The Bäcklund transformation for the sine-Gordon equation (2.6) is

$$\frac{1}{2}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)(u - u') = a \sin\left(\frac{1}{2}(u + u')\right) \quad , \tag{4.24a}$$

$$\frac{1}{2}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)(u + u') = -\frac{1}{a} \sin\left(\frac{1}{2}(u - u')\right) \ , \tag{4.24b}$$

with an arbitrary constant a. We introduce a function defined by

$$\Gamma = \tan\left(\frac{1}{4}(u + u')\right) \quad , \tag{4.25}$$

and eliminate u' in eqs.(4.24a ~ b). We get

$$\Gamma_t + \Gamma_x = -a\,\Gamma + \frac{1}{2}\,w\,(1 + \Gamma^2) \ , \tag{4.26a}$$

$$\Gamma_t - \Gamma_x = \frac{1}{a}\,\Gamma\cos u - \frac{1}{2a}(1 - \Gamma^2)\sin u \quad , \tag{4.26b}$$

where

$$w = u_t + u_x \quad . \tag{4.27}$$

Defining

$$\Gamma = \frac{1}{i}\frac{\psi_2 + i\,\psi_1}{\psi_2 - i\,\psi_1} \quad \text{and} \quad a = -4i\,\lambda\,, \tag{4.28}$$

we obtain coupled linear equations for ψ_1 and ψ_2 ,

$$\begin{cases} \dfrac{\partial}{\partial x}\psi_1 + \dfrac{i}{4}w\psi_1 + \dfrac{1}{16\lambda}\exp(-iu)\psi_2 = \lambda\,\psi_2\,, \\[2ex] -\dfrac{\partial}{\partial x}\psi_2 + \dfrac{i}{4}w\psi_2 + \dfrac{1}{16\lambda}\exp(iu)\psi_1 = \lambda\,\psi_1\,, \end{cases} \tag{4.29a}$$

$$\begin{cases} \dfrac{\partial}{\partial t}\psi_1 = \lambda\psi_2 - \dfrac{i}{4}w\psi_1 + \dfrac{1}{16\lambda}\exp(-iu)\psi_2 = \dfrac{\partial}{\partial x}\psi_1 + \dfrac{1}{8\lambda}\exp(-iu)\psi_2\,, \\[2ex] \dfrac{\partial}{\partial t}\psi_2 = -\lambda\psi_1 + \dfrac{i}{4}w\psi_1 - \dfrac{1}{16\lambda}\exp(iu)\psi_1 = \dfrac{\partial}{\partial x}\psi_2 - \dfrac{1}{8\lambda}\exp(iu)\psi_1\,. \end{cases} \tag{4.29b}$$

These are the fundamental equations of the inverse scattering method for the sine-Gordon equation.[28] In general, the inverse scattering method can be derived from the Bäcklund transformation by linearizing the equations with respect to auxiliary functions which are introduced to relate a solution to the other.[27]

4.3). The conservation laws.

It has been known that the soliton system has an infinite number of conservation laws,

$$\frac{\partial}{\partial t} D_i[u] + \frac{\partial}{\partial x} F_i[u] = 0 \quad , \quad i = 1, 2, \cdots \tag{4.30}$$

wehre $D_i[u]$ and $F_i[u]$ are called conseved density and conserved flux, respectively. For instance, the first three conserved densities and fluxes of the Korteweg-de Vries equation are

$$D_1 = u \quad , \qquad F_1 = -3\,u^2 + u_{xx} \, ,$$

$$D_2 = \frac{1}{2}\,u^2 \quad , \qquad F_2 = -2\,u^3 + u\,u_{xx} - \frac{1}{2}\,u_x^2 \quad , \tag{4.31}$$

$$D_3 = \frac{1}{3}u^3 + \frac{1}{6}\,u_x^2 \quad ,$$

$$F_3 = -\,\frac{3}{2}\,u^4 + u^2 u_{xx} - 2u\,u_x^2 + \frac{1}{3}u_x\,u_{xxx} - \frac{1}{6}\,u_{xx}^2 \; .$$

Here, we present a systematic method to derive an infinite number of conservation laws from the formalism of the inverse scattering method.[27),29)] We start form the formalism developed by Ablowitz et al. Using a function defined by $\Gamma = \psi_2/\psi_1$, eqs.(4.15) and (4.16) are rewritten as

$$q\Gamma = \frac{1}{2\eta}\,[rq - (q\Gamma)^2 - q\,\frac{\partial}{\partial x}\,(q\Gamma/q)] \quad , \tag{4.32a}$$

$$\frac{\partial}{\partial t}(q\Gamma) = \frac{\partial}{\partial x}\,(A + B\Gamma) \; . \tag{4.32b}$$

We notice eq.(4.32b) is in the form of conservation laws. Substitution of

$$q\;\Gamma = \sum_{n=1}^{\infty} f_n \;\eta^{-n} \quad , \tag{4.33}$$

into eq.(4.32a) yields a recursion formula for f_n;

$$f_{n+1} = \frac{1}{2}[\,(rq)\,\delta_{n0} - \sum_{k=1}^{n-1} f_k\,f_{n-k} - q(\frac{f_n}{q})x] \quad . \tag{4.34}$$

Therefore, when the given nonlinear evolution equation is expressed in the formalism of Ablowitz et al. (namely, if explicit forms of r, q, A and B are given), an infinite sequence of the conservation laws are obtained in a simple way. The existence of an infinite number of the conservation laws in the soliton system implies that the system is an completely integrable system. The proof of completely integrability of soliton system has been given by several authors.[30)]

The above brief survey aimes to provide general impression on the recent advancement of analytical methods of solving nonlinear evolution equations. Though the problem is one of the hardest, the progress has been steadily accelerated since the first ingenious discovery of the inverse scattering method by Gardner, Greene, Kruskal and Miura. In particular, since inter-relationships among the various methods of solving the nonlinear evolution equations have been clearly understood, we can expect studies of the nonlinear physics will be developed in a systematic way on the firm mathematical ground.

§5. Dynamical Properties of Dressed Solitons

We have seen in the previous section that now very powerful arms are available to deal with the nonlinear evolution equations such as the Korteweg-de Vries equation and the nonlinear Schrödinger equation. These nonlinear evolution equations, however, are nothing but approximate model equations for real physical systems, derived in the lowest order expansion of the reductive perturbation theory. So far we have discussed the analysis of its lowest order, but nonlinear evolution equation.

Now, from the point of view of perturbation approaches, we have undertaken investigation of the higher order contributions of the reductive perturbation expansion.[31)] Referring back to the example of the nonlinear ion-acoustic wave in collisionless plasma discussed in section 3, we ask how the second order potential $\psi^{(2)}$ behaves when the first order potential $\psi^{(1)}$ is determined by the Korteweg-de Vries equation Expanding the set of eqs.(3.15a ~ c) into power series of ε, we have obtained the Korteweg-de Vries equation for the first order potential in the second order expansions, and then the second order quantities $n^{(2)}$ and $u^{(2)}$ are expressed as

$$n^{(2)} = \psi^{(2)} + \frac{1}{2}\psi^{(1)}\psi^{(1)} - \frac{\partial^2}{\partial\xi^2}\psi^{(1)} , \tag{5.1a}$$

$$u^{(2)} = \psi^{(2)} - \frac{1}{2}\frac{\partial^2}{\partial\xi^2}\psi^{(1)} . \tag{5.1b}$$

These relationships correspond to eq.(3.17) for the first order quantities. In order to determine behavior of the second order potential $\psi^{(2)}$, we have to proceed up to the third order terms. Then, as the compatibility condition among three equations in the third order, we obtain an equation for the second order potential $\psi^{(2)}$ as

$$\frac{\partial}{\partial\tau}\psi^{(2)} + \frac{1}{2}\frac{\partial^3}{\partial\xi^3}\psi^{(2)} + \frac{\partial}{\partial\xi}(\psi^{(1)}\psi^{(2)}) = S(\psi^{(1)}) , \tag{5.2}$$

where

$$S(\psi^{(1)}) = -\frac{3}{8}\frac{\partial^5}{\partial\xi^5}\psi^{(1)} + \frac{1}{2}\psi^{(1)}\frac{\partial^3}{\partial\xi^3}\psi^{(1)} + \frac{5}{8}\frac{\partial}{\partial\xi}\left(\frac{\partial\psi^{(1)}}{\partial\xi}\right)^2 . \tag{5.3}$$

Thus, the Korteweg-de Vries equation (3.18) and the linear inhomogeneous equation (5.2) with (5.3) describes a nonlinear ion acoustic wave propagation in the second order approximation. Differing from the ordinary perturbation expansion, we observe that the reductive perturbation theory provides the lowest order equation in which essential nonlinear effect is accounted for completely, while the second order equation describes interaction effects between the fundamental nonlinear wave and the higher order dispersion effects.

Firstly, seeking a type of solutions $\psi^{(1)}(\eta)$ and $\psi^{(2)}(\eta)$ with argument

$$\eta = \xi - \lambda\tau \quad , \tag{5.4}$$

we have obtained a steady one soliton solution of the coupled set of eqs. (3.17) and (5.2) with (5.3) as follows,

$$\psi^{(1)}(\eta) = 3\lambda \ \operatorname{sech}^2(D\eta) \quad , \tag{5.5a}$$

$$\psi^{(2)}(\eta) = \frac{4}{9}\lambda^2\operatorname{sech}^2(D\eta)\ \{2D\eta \tanh(D7) - 8 + 7 \operatorname{sech}^2(D\eta)\} \tag{5.5b}$$

with

$$D = (\lambda/2)^{1/2} \quad . \tag{5.6}$$

The perturbed potential calculated up to the second order

$$\psi(\eta) = \psi^{(1)}(\eta) + \psi^{(2)}(\eta) \tag{5.7}$$

can be regarded as the dressed soliton, of which velocity λ is given by the amplitude A of the ion accoustic potential perturbation as

$$\lambda = \frac{1}{3} A + \frac{1}{12} A^2 \quad . \tag{5.8}$$

We have observed numerically that the steady state clouds (5.5b) moves stably with the Korteweg-de Vries soliton core (5.5a).

Secondly, the collision processes of the dressed solitons attract our special interests.[32] As a solution of the Korteweg-de Vries equation (3.17), we now take the well-known two-soliton solution,

§6. Concluding Remarks

As we discussed above, the study of nonlinear waves stemmed from the observation of solitary wave on a canal in the middle of the 19th century has made a remarkable stride in the last decade. Intense researches on nonlinear waves occuring in the various field of physics has provided us new physical concepts such as solitons and envelop solitons and also very powerful mathematical tools such as the reductive perturbation theory, the inverse scattering method, the Bäcklund transformation. We devoted a large part of the present paper to the discussion on nonlinear evolution equations in the following reasons. For a long time,"linear" view point of physics based on Fourier analysis even govern our way of thinking. The buds of new physical picture "soliton" originated from numerical analysis of the Korteweg-de Vries equation has been secured very firmly by advancement of the mathematical methods. Development of the analytical method of solving full variety of the nonlinear evolution equations has conversely established various kinds of solutions as discussed in section 2. Enduring studies on the nonlinear evolution equation will shed bright light on the fields of nonlinear physics.

So far, the theories of soliton have been mainly dialing with one-dimensional and classical nonlinear waves. Multi-dimensional soliton and its quantum theory are no doubt challenging subjects. It is almost certain that further studies will introduce modifications and extensions of the concepts and the methods now in our hands. However, it is to be noted that even one-dimensional and classical theory of soliton has clarified lots of phenomena which could not be handled within the frame work of linearized theory.

As an illustration of slight extension of the approach, we have discussed new notion of dressed soliton and dressed envelop soliton on the basis of the higher order analysis of the reductive perturbation theory. Systematic studies of interaction effects among the nonlinear waves and between the particles and the nonlinear waves will help us to construct physical soliton picture in the real world.

Very recently, there has been a tremendous number of papers entitled "soliton" in physics of elementary particles and field theory.[34] Concerned subjects are quantization of soliton, vortex solution, monopole solution, topological quantum number, extended system, quark confinement, spontaneous symmetry breakdown, etc. Each notion has its own history and motivation. Some of them are very close to the concept of soliton and some are not. All of them, however, are based on the full recognition that nature is essentially nonlinear.

The authors believe the study of soliton is the first successful attempt to establish the systematic understanding of the truth of nonlinear phenomena. Then, as a concluding remark of the present paper, it will not be an overcharge to say; "we are facing the dawn of nonlinear physics."

We feel it a great honor to dedicate this paper to Professor Ta-You Wu in celebration of his seventieth birthday.

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Caption of Figures

Fig. 1 A Soliton

Fig. 2 A kink

Fig. 3a A bright envelop soliton for the case of $pq > 0$

Fig. 3b A dark envelop soliton for the case of $pq < 0$

Fig. 4 Collision process of two solitons

Fig. 5 Collision of two triple solitary pulses $\lambda_i (i=1,2)$ and $\gamma_i^{-1} (i=1,2)$ represent the velocity and the width of each triple solitary pulse, respectively

Fig. 6 Mechanical analogue of the sine-Gordon equation. Angle ψ measure a twist in (z,y) plane.

Fig. 7 Temporal evolution of shape of two dressed solitons in collision process. The thick lines represent the dressed solitons ψ , the thin lines the Korteweg-de Vries´ soliton core $\psi^{(1)}$, the broken lines the clouds $\psi^{(2)}$, respectively.

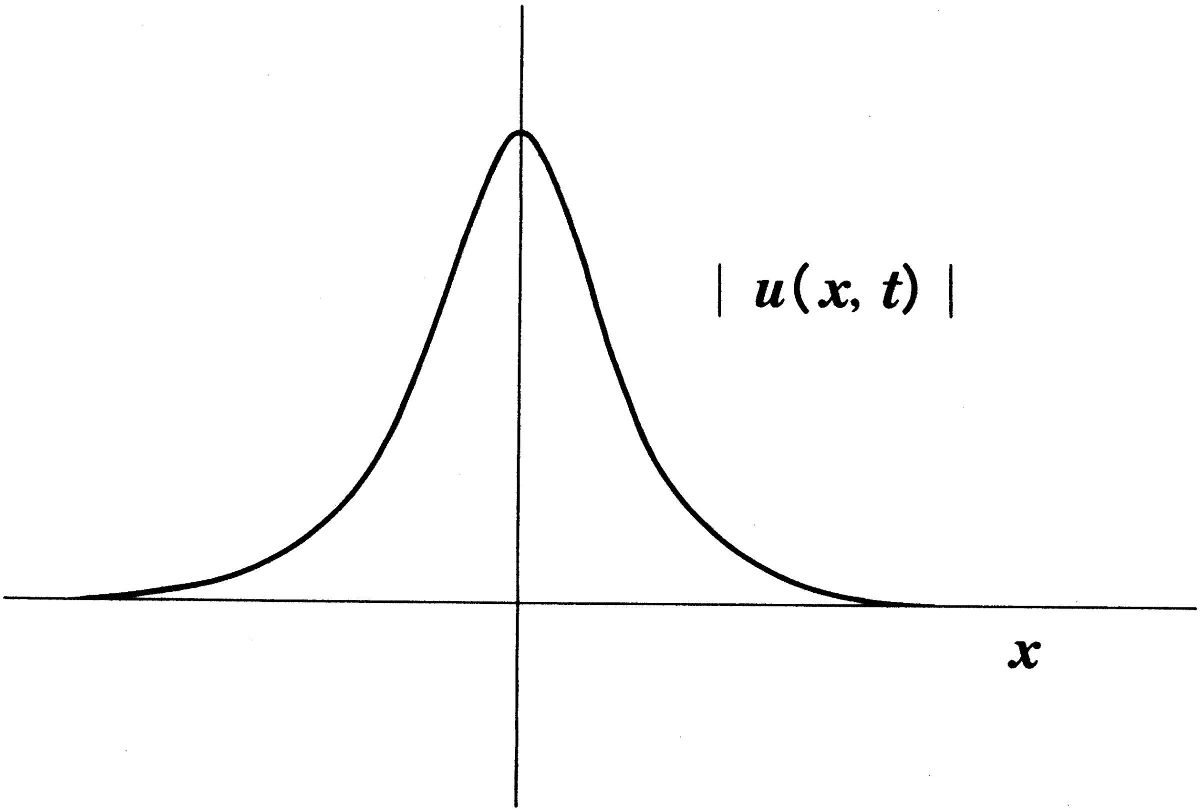


FIG. 1

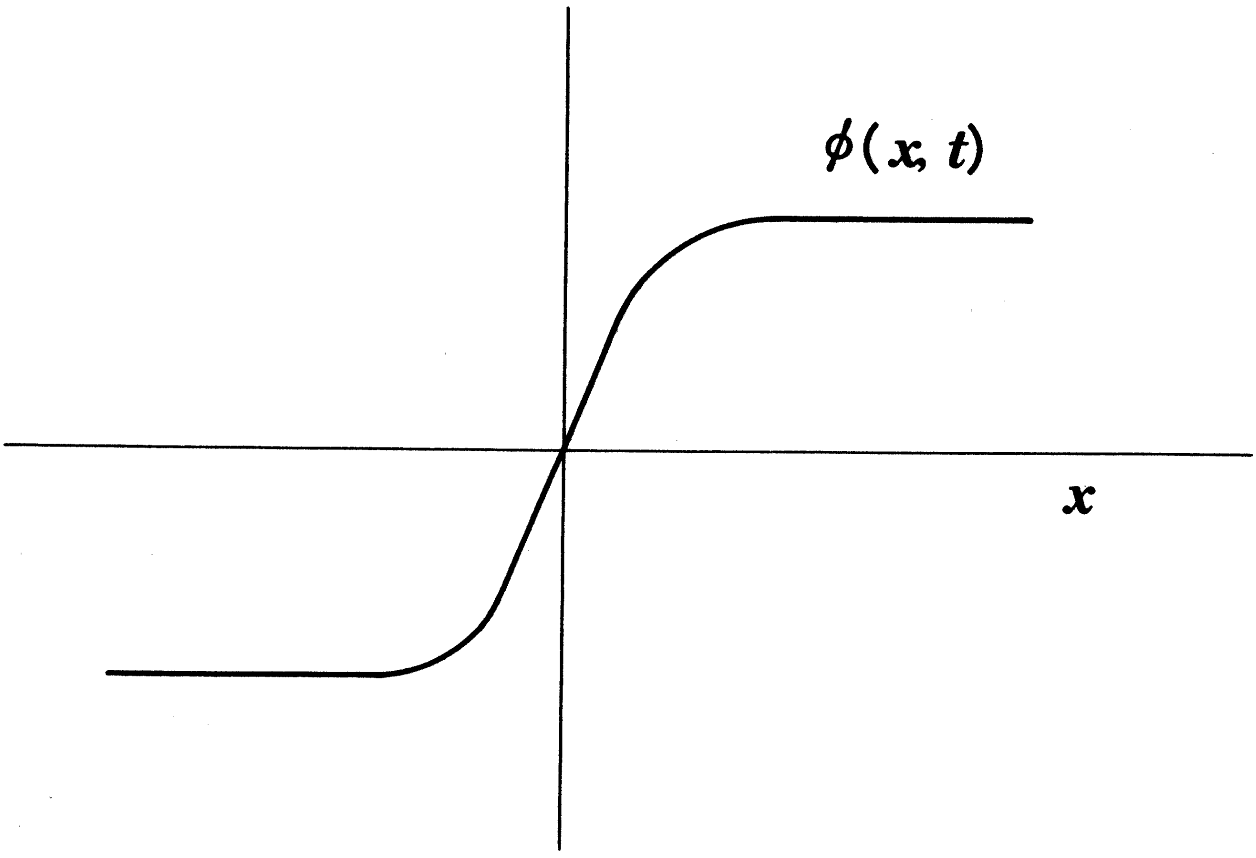


FIG. 2

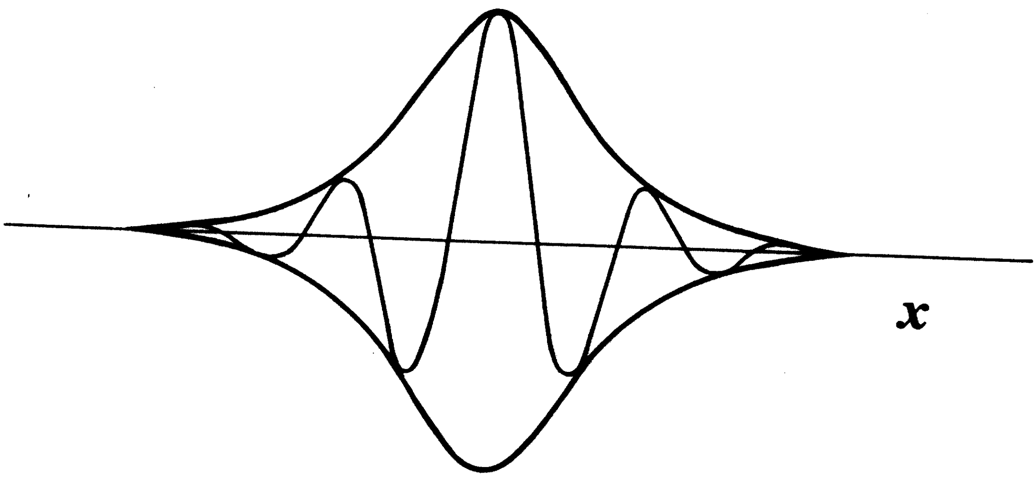


FIG. 3A

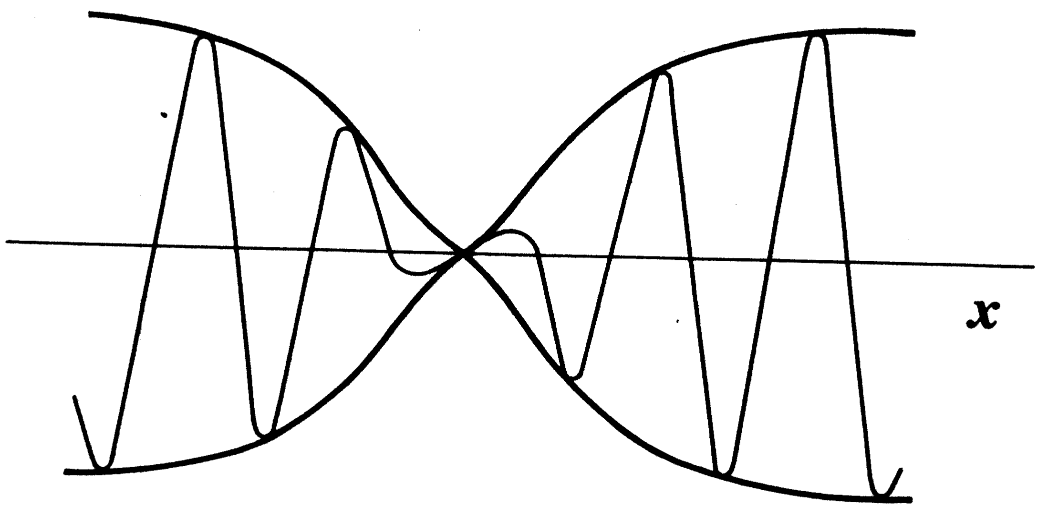


FIG. 3B

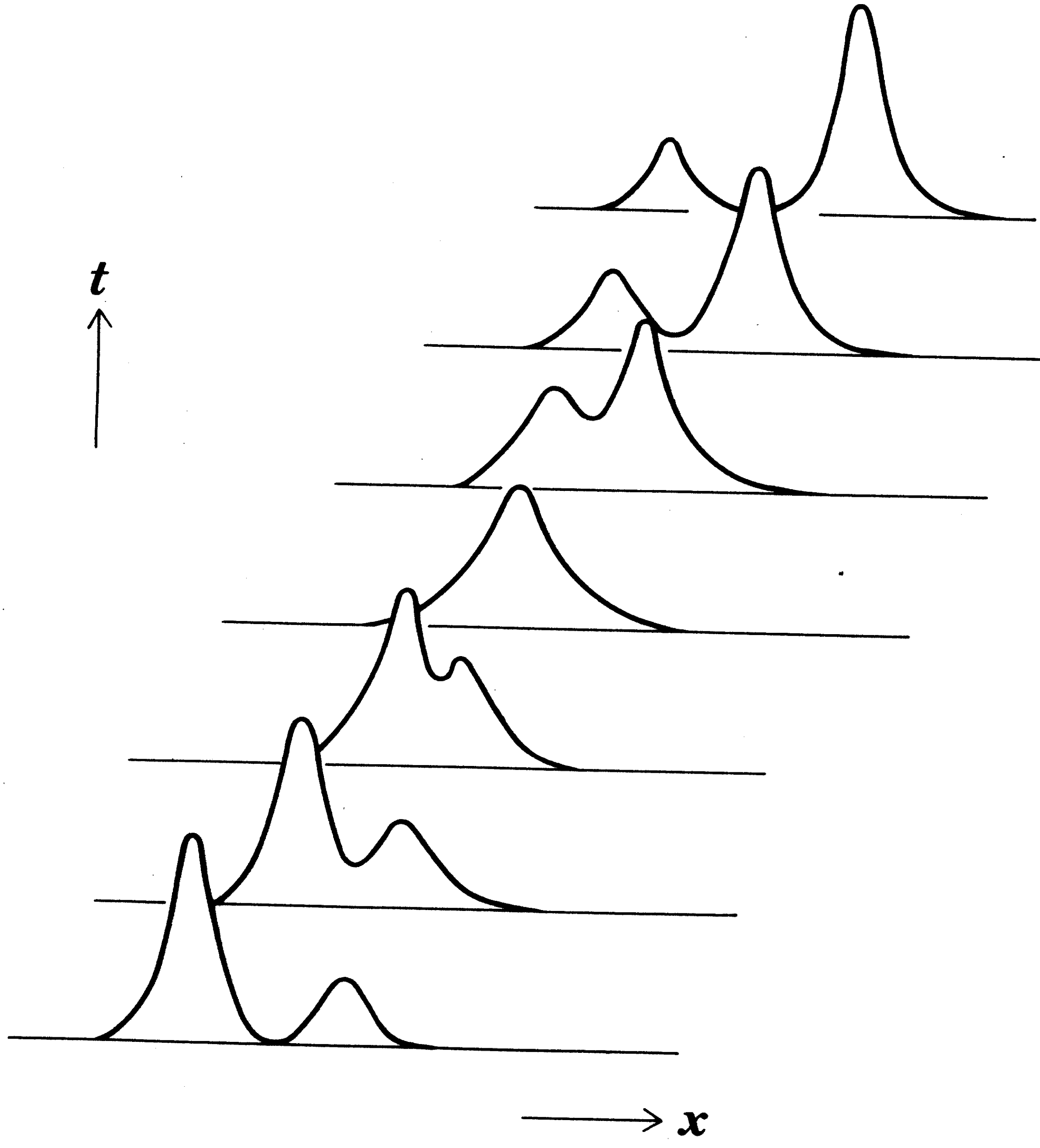
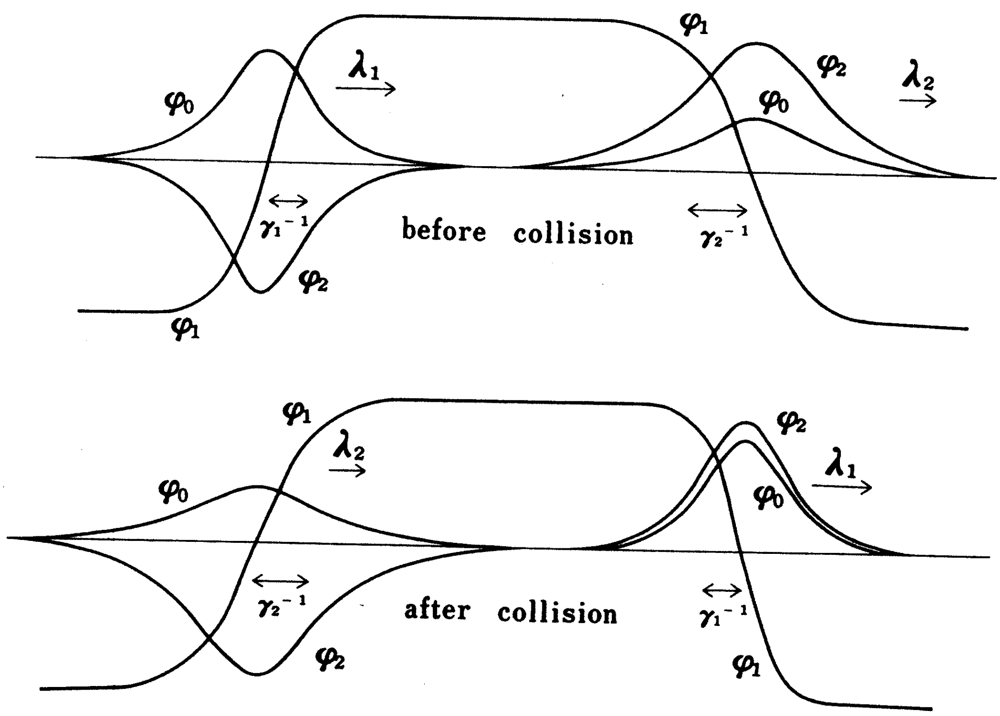
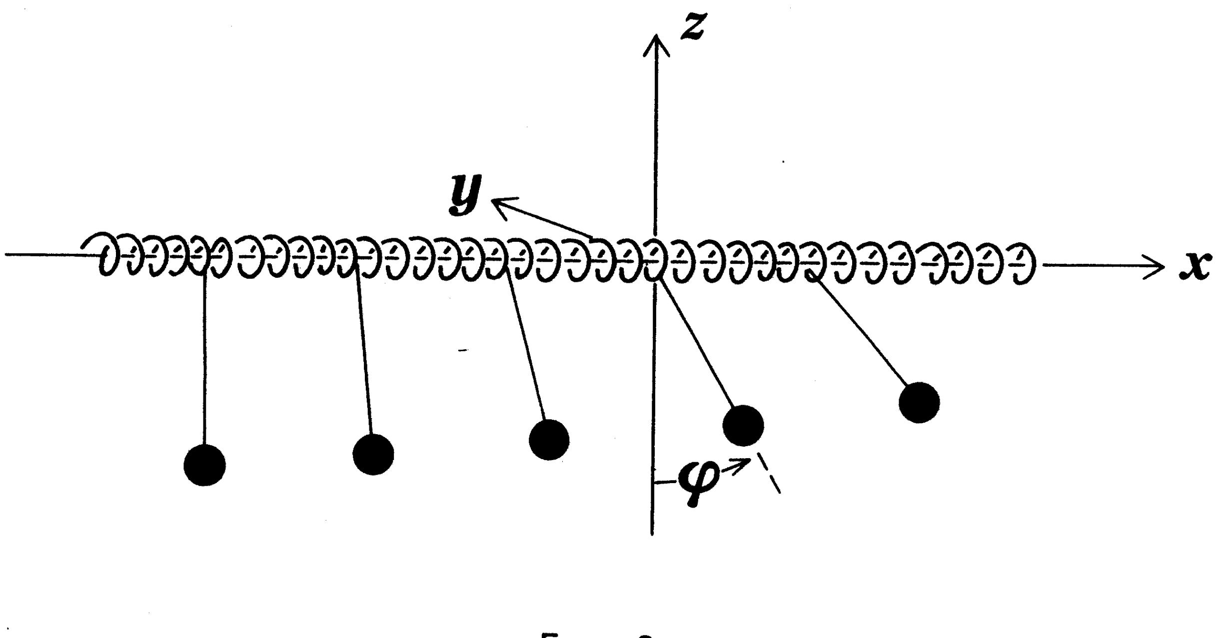


FIG. 4



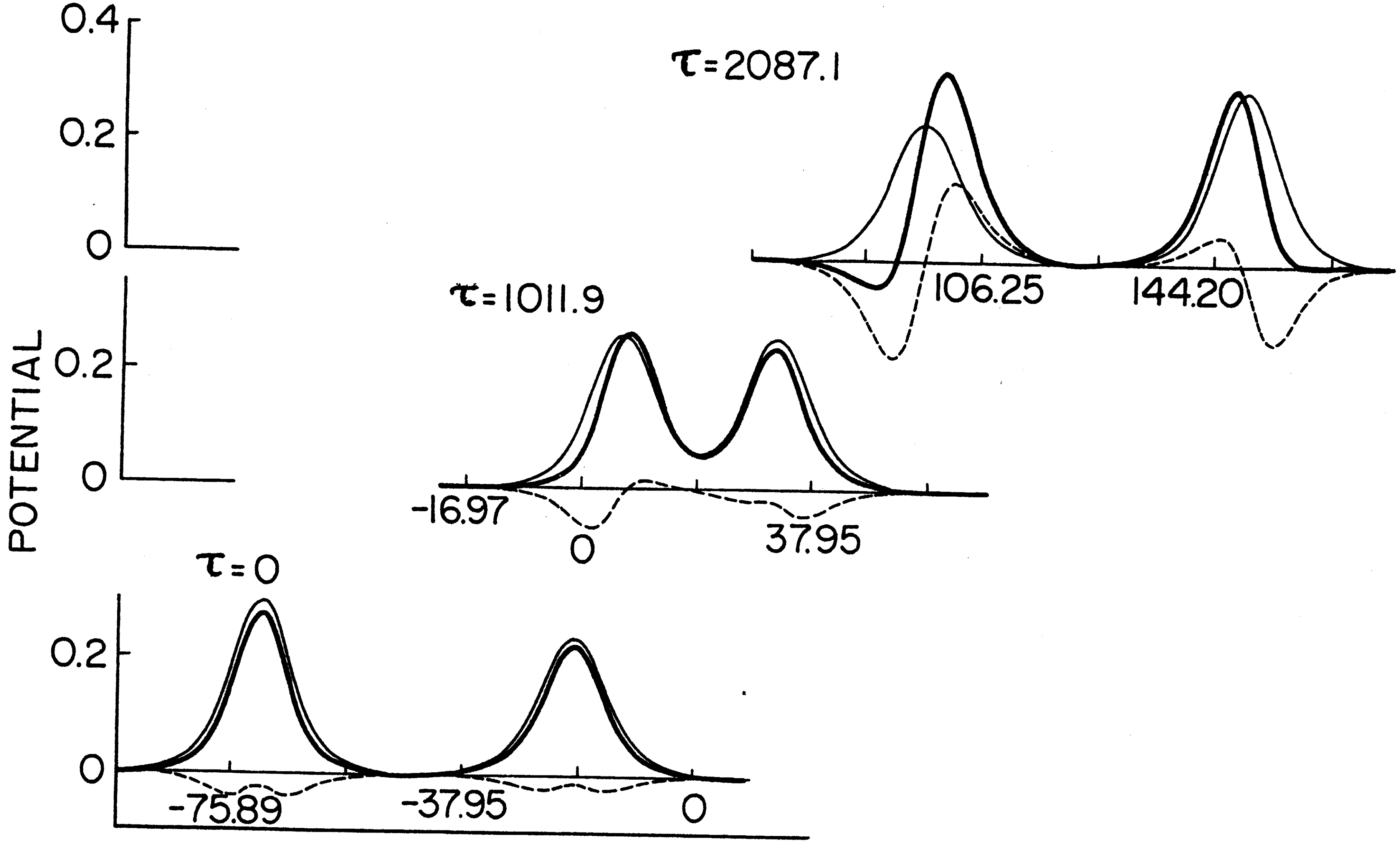
φ_1
λ_1
φ_0
φ_2
λ_2
φ_0
γ_1^{-1}
before collision
γ_2^{-1}
φ_2
φ_1
φ_1
λ_2
φ_2
φ_0
λ_1
φ_0
γ_2^{-1}
after collision
γ_1^{-1}
φ_2
φ_1

FIG. 5



z
y
x
φ

FIG. 6



0.4
0.2
0
τ=2087.1
106.25
144.20
τ=1011.9
POTENTIAL
0.2
0
-16.97
0
37.95
τ=0
0.2
0
-75.89
-37.95
0

Fig. 7

Errata of the manuscript of Ichikawa and Wadati,

"Solitons in Plasma and Other Dispersive Media."

p.3	1st line, from the bottom	interactions * waves	→ interaction among waves
p.14	10th line, from the top	compativility*	→ compatibility
p.15	9th line, from the top	eigen * vector	→ eigenvector
p.20	1st line, from the bottom	$= \overset{*}{C}_n(t)\exp(\kappa_n x)$	→ $=c_n(t)\exp(\kappa_n x)$
p.23	1st ∿ 3rd line from the top	$= q\overset{*}{c} - rB$	→ $= qC - rB$
		$- 2\overset{*}{7}B =$	→ $- 2\eta B$
		$+ 2\overset{*}{7}C =$	→ $+ 2\eta C$
p.24	14th line, from the top	$-w')\overset{*}{f}w_{xx}-w'_{xx})$	→ $-w')(w_{xx}-w'_{xx})$
p.24	3rd line, from the bottom	$- 2\eta\ \tan{*}h(\eta\xi)$	→ $- 2\eta\ \tanh(\eta\xi)$
p.25	10th line, from the top	$\frac{1}{2}(w_0 - \underset{*}{w_1})^2$	→ $\frac{1}{2}(w_0 - w_2)^2$
p.27	3rd line, from the bottom	wehre*	→ where
p.28	7th line, from the bottom	$-q(\frac{f_n}{q})\overset{*}{x}$	→ $-q(\frac{f_n}{q})_x$
p.30	3rd line, from the bottom	equation* Expanding	→ equation. Expanding
p.34	7th line, from the bottom	dialing*	→ dealing
p.36	ref. 8)	M.Toda; Prog. Teort*	→ M.Toda; Prog. Theoret.
	ref.18)	Prog. Theoret*	→ Prog. Theoret.
p.37	ref.22)	and A.B. Schbat*	→ and A.B. Shbat
	ref.25)	Teor. Fiz.* (Soviet	→ Teor. Fiz. 67, 2131 (Soviet
	ref.28)	L.D. Faddev*	→ L. D. Faddeev
p.38	ref.34)	J.I.* Gervais	→ J. L. Gervais
p.39	Fig. 5	solitary pulses*	→ solitary pulses.

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* represents errata