INSTITUTE OF PLASMA PHYSICS

NAGOYA UNIVERSITY

RESEARCH REPORT

NAGOYA, JAPAN

DC Current in the Collisionless Limit Induced by a Travelling Wave - A calculation Based on Collision Term

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IPPJ - 284 March 1977

Futher communication about this report is to be sent to the Research Information Center, Institute of Plasma Physics, Nagoya University, Nagoya 464, Japan

Abstract

The DC current induced by a Travelling Wave is calculated on the basis of the assumption that the distribution function of electrons in the collisionless limit should be determined by a condition derived from the nature of the collision operator, as in the case of the calculation of the neoclassical transport in a torus. The resultant net current is found to have the same parameter dependence as the one derived in a previous analysis, in which we assumed the initial distribution of electrons to be uniform and isotropic Maxwellian. The numerical coefficient is found, however, to be a little different from the previous one. The importance of the accurate matching of the distribution function of untrapped particles to the Maxwellian one for large velocity is demonstrated.

§1. Introduction

Travelling wave is a promising means to drive DC current necessary for the steady operation of a Tokamak fusion reactor.

In a previous paper[1), one of the authors calculated the DC current induced in a magnetized plasma by a travelling wave. There it was assumed that the electrons are perfectly collisionless and the initial velocity distribution function of electrons is a uniform, isotropic Maxwellian. Since, however, these assumptions seem a little artificial, it is desirable to set a more natural, physical basis for the calculation.

In a well-known paper on the damping of a plasma wave[2)], Zakharov and Karpman showed that, in a plasma with very rare collisions, the collisionless limit of the distribution function can be determined by an analysis, in which the collision term plays an essential role. This idea forms also the bases of the analyses on the neoclassical transport in tokamaks.[3,4)] Accordingly, the same idea should present us a sound physical basis for the calculation of the DC current induced by a travelling wave.

In this paper, we shall calculate the DC current by the use of the idea stated above. The result is found to have the same parameter-dependence as that derived in the previous paper[1)], but the numerical coefficient is somewhat different. In the course of calculation, it was found that the distribution function of untrapped particles in our analysis must be matched to the Maxwellian one at a large velocity more accurately than done in the previous papers.[2,3)] The implication of the same kind of modification for other problems will be discussed in separate papers.

§2. Model

The model is almost the same as the one in the previous paper[1]. We consider a plasma in a steady magnetic field, which is so strong that in the absence of collisions each electron is completely frozen to a certain magnetic line of force. The travelling wave provides a modification of the magnetic feild. In the laboratory system the total magnetic field is of the form: $B_0\{1+\varepsilon\cos(kz - \omega t)\}$.

In the followings, we shall make the analysis in the wave frame, in which the magnetic field seems to be static:

$$B(z) = B_0 \{1 - \varepsilon \cos kz\} \quad . \tag{2.1}$$

The motion of an electron is adiabatic, so that the magnetic moment

$$\mu = \frac{mv_\perp^2}{2B(z)} \tag{2.2}$$

is a constant of motion, where $\vec{v}_\perp$ is the velocity perpendicular to the z axis and m is the electron mass. If collisions are neglected, the equation of motion for an electron is given by

$$m \frac{dv_z}{dt} = - \mu \frac{\partial}{\partial z} B(z) = - \mu B_0 \varepsilon k \sin kz \quad . \tag{2.3}$$

This equation has the energy integral

$$\frac{1}{2} m v_z^2 + \mu \varepsilon B_0 (1 - \cos kz) = E > 0 \quad . \tag{2.4}$$

It is convenient to introduce new variables ξ , κ and σ in place of $v_\perp$ and v_z by

$$\left.\begin{aligned} \xi &= 2\mu B_0/(m v_T^2) , \\ \kappa^2 &= E/(2\varepsilon\mu B_0) , \\ \sigma &= \mathrm{sign}\ (v_z) , \end{aligned}\right\} \tag{2.5}$$

where $v_T = (2T_e/m)^{1/2}$ is the electron thermal velocity. In terms of these new variables, v_z is expressed as

$$v_z = \varepsilon^{1/2} v_T \alpha \sqrt{2\xi}(\kappa^2 - \sin^2 \frac{kz}{2})^{1/2} . \tag{2.6}$$

It is easily seen that $\kappa > 1$ for an electron trapped in the wave and $\kappa < 1$ for an untrapped electron. Note that all the new variable ξ , κ and σ are constant of motion in the absence of collisions.

§2. Kinetic Equation and Its Solution

The kinetic equation for electrons in a steady state is given by

$$v_z \frac{\partial f_e}{\partial z} - \frac{\mu}{m} \frac{\partial B}{\partial z} \frac{\partial f_e}{\partial v_z} = C_e(f_e) , \tag{3.1}$$

where $f_e(\vec{v}, z)$ is the distribution function of electrons and $C_e(f_e)$ represents effects of collisions. If we assume the problem to be axisymmetric and use ξ , κ , σ as the new independent variables in place of $\vec{v}$, eq.(3.1) is reduced to

$$v_z(\xi,\kappa,\sigma, z) \frac{\partial f_e}{\partial z} = C_e(f_e) . \tag{3.2}$$

The collision term contains contributions of electron-electron collisions and electron-ion collisions. In dealing with collisions with ions, we take the distribution function of ions to be Maxwellian in the laboratory frame. As for the electron-electron collisions, we linearize the collision term by assuming that deviations from the Maxwellian distribution are small and important only in small region of velocities of resonant particles. We further assume that the distribution function is most sensitive to changes in the longitudinal velocity, so that in the first approximation we can neglect all the other derivatives. Then the collision term assumes the form[3)]

$$C_e(f_e) = \nu \frac{2v_T}{\varepsilon\xi} (\kappa^2 - \sin^2 \frac{kz}{2})^{1/2} \frac{\partial}{\partial\kappa^2} \{A(\vec{v}) [(\kappa^2 - \sin^2 \frac{kz}{2})^{1/2} \times (\frac{\partial f_e}{\partial\kappa^2} + 2\varepsilon\xi f_e) + \sigma\sqrt{2\varepsilon\xi}(v_{ph}/v_T) f_e]\} \quad , \tag{3.3}$$

where $v_{ph}=\omega/k$ is the phase velocity of the wave, $\nu=2\pi\lambda n_0 e^4/(m^2 v_T^3)$ is a measure of the collision frequency (λ being the Coulomb logarithm) and the coefficient $A(\vec{v})$ is given by the following expression:

$$A(\vec{v}) = \{1+(1-\frac{1}{2s})\eta(s) + \frac{d\eta(s)}{ds}\} \frac{w_\perp^2}{w^3} + \frac{w_z^2}{w^3} \frac{\eta(s)}{s} ,$$

where $\eta(s) = \frac{2}{\sqrt{\eta}} \int_0^s e^{-t} \sqrt{t}\, dt$, $\vec{w} = (v_x, v_y, v_z + v_{ph})$ and $s = w^2/v_T^2$.

We solve eq.(3.2) by assuming that ν is small and writing

$$f_e = f_0 + f_1 + \cdots\cdots \quad , \tag{3.4}$$

where f_0 is the distribution function in the collisionless limit and f_1 is the correction term proportional to ν . Then eq.(3.2) reduces to

$$\frac{\partial f_0}{\partial z} = 0 \quad , \tag{3.5}$$

$$v_z(\xi,\kappa,\sigma,\ z)\ \frac{\partial f_1}{\partial z} = C_e(f_0) \quad , \tag{3.6}$$

and so on. Equation (3.5) simply states that f_0 is independent of z:

$$f_0 = f_0(\xi,\ \kappa,\ \sigma) \quad . \tag{3.7}$$

Equation (3.8) is rewritten as

$$\frac{\partial f_1}{\partial z} = \nu\ \frac{1}{\sqrt{2\varepsilon^3\xi}}\ \frac{\partial}{\partial\kappa^2}\ \{A(\vec{v})\,[\sigma(\kappa^2-\sin^2\frac{kz}{2})^{1/2}\ (\frac{\partial f_0}{\partial\kappa^2} + 2\varepsilon\xi f_0)$$

$$+ \sqrt{2\varepsilon\xi}\ \ \alpha f_0\ \} \quad , \tag{3.8}$$

where $\alpha = v_{ph}/v_T$.

The coefficient $A(\vec{v})$ has a weak dependence on z if we regard it as a function of ξ, κ, σ and z. Since, however, this dependence is higher order in ε, we can neglect it. Then the z-dependence of the right-hand side of eq.(3.8) comes solely from the expression $(\kappa^2 - \sin^2\frac{kz}{2})^{1/2}$. We further assume that the whole problem is periodic in z with the period $2\pi/k$.

For untrapped particles, we integrate (3.8) with respect to z over a period:

$$\frac{\partial}{\partial\kappa^2}\ \{A[J(\kappa)\,(\frac{\partial f_0}{\partial\kappa^2} + 2\varepsilon\xi f_0) + \sigma\alpha\sqrt{2\varepsilon\xi}\ f_0]\} = 0 \quad , \tag{3.9}$$

where

$$J(\kappa) = \frac{k}{2\pi}\int_0^{\frac{2\pi}{k}} (\kappa^2 - \sin^2\frac{kz}{2})^{1/2}\ dz$$

$$= \frac{2}{\pi}\ \kappa E(1/\kappa) \quad , \tag{3.10}$$

$E(x)$ being the complete elliptic integral of the second kind. Equation (3.10) has a solution of the form:

$$f_0 = C \exp \{-2\varepsilon\xi\kappa^2 - \sigma\sqrt{2\varepsilon\xi}\ \alpha \int_1^{\kappa^2} \frac{d\kappa^2}{J(\kappa)}\} \quad \text{for } \kappa > 1 . \quad (3.11)$$

The constant C should be chosen so that when $\kappa >> 1$, f_0 should tend to the Maxwellian distribution:

$$f_M(\vec{v}) = (\frac{1}{\pi v_T^2})^{3/2} \exp(-\frac{v_\perp^2 + (v_z+v_{ph})^2}{v_T^2})$$

$$= (\frac{1}{\pi v_T^2})^{3/2} e^{-\alpha^2} \exp[-(\xi-\varepsilon\xi+2\varepsilon\xi\kappa^2 + 2\sigma\alpha\sqrt{2\varepsilon\xi}\ (\kappa^2 - \sin^2\frac{kz}{2})^{1/2})]$$

$$= (\frac{1}{\pi v_T^2})^{3/2} e^{-\alpha^2} \exp[-(\xi-\varepsilon\xi + 2\varepsilon\xi\kappa^2 + 2\sigma\alpha\sqrt{2\varepsilon\xi}\ \kappa + O(1/\kappa)] \quad (3.12)$$

On the other hand, the integral in eq.(3.11) has the following asymptotic expression for large κ:

$$\frac{1}{2}\int_1^{\kappa^2} \frac{d\kappa^2}{J(\kappa)} = \frac{\pi}{4}\int_1^{\kappa^2} \frac{dt}{t^{1/2}E(t^{-1/2})}$$

$$= \kappa - c_0 - \frac{1}{4\kappa} - \frac{7}{192}\frac{1}{\kappa^3} + \cdots \quad \text{for } \kappa >> 1, (3.13)$$

where

$$c_0 = 1 - \int_0^1 \frac{1}{u^2}(\frac{\pi}{2E(u)} - 1)du$$

$$= 0.6894 \cdots \quad (3.14)$$

Comparing eqs.(3.11) and (3.12), we have for $\kappa > 1$

$$f_0(\xi, \kappa, \sigma) = (\frac{1}{\pi v_T^2})^{3/2} e^{-\alpha^2}$$

$$\times \exp\{-(\xi-\varepsilon\xi+2\varepsilon\xi\kappa^2 + 2\sigma\alpha\sqrt{2\varepsilon\xi}\ F(\kappa))\} , \quad (3.15)$$

where

$$F(\kappa) = \frac{\pi}{4}\int_1^{\kappa^2} \frac{dt}{t^{1/2}\ E(t^{-1/2})} + c_0 \quad . \tag{3.16}$$

The importance of the presence of the constant term c_0 will be demonstrated in §5.

For the trapped particles, $f_0(\xi,\kappa,\sigma)$ should be independent of σ, because otherwise no steady state is possible.[5] Therefore, in the collisionless limit, the contribution of trapped particles to the electron flow vanishes in the wave frame.

§4. DC Current

Knowing the distribution function $f_0(\vec{v})$, we can calculate the DC current density:

$$j_e = - en_0 \int f_0(\vec{v})\ v_z\ d\vec{v} \quad , \tag{4.1}$$

where n_0 is the average electron density. In terms of the variables ξ, κ and σ, v_z is expressed by eq.(2.6) and $v_\perp{}^2$ is given by

$$v_\perp{}^2 = \xi\,(1 - \varepsilon \cos kz)\ v_T{}^2 \ . \tag{4.2}$$

The volume element in the velocity space, therefore, can be written as

$$2\pi v_\perp dv_\perp dv_z = v_T{}^3\ \pi\sqrt{\frac{\varepsilon\xi}{2}}\ \frac{1 - \varepsilon \cos kz}{(\kappa^2 - \sin^2\frac{kz}{2})^{1/2}}\ d\kappa^2\ d\xi\ \sum_\sigma \ . \tag{4.3}$$

We should note here that since the magnetic field has the z-dependence of the form eq.(2.1), the cross-section of a magnetic flux tube is proportional to $1/(1 - \varepsilon \cos kz)$.

Taking this into account and using eqs.(2.6) and (4.3), we obtain the DC current carried by electrons:

$$I_e = - en_0 S_0 \ \pi \sum_\sigma \int d\kappa^2 \int d\xi \ \sigma \ \varepsilon \ \xi \ v_T^4 f_0(\xi,\kappa,\sigma) \ , \qquad (4.4)$$

where S_0 is the cross-section of plasma at $kz=\pi/2$. The trapped particles give no contribution to this integral, because their distribution function is independent of σ . Then, substituting eq.(3.15) into eq.(4.4), we have

$$I_e = - e \ n_0 \ S_0 \frac{v_T}{\sqrt{\pi}} \exp(-\alpha^2) \sum_\sigma \int_1^\infty d\kappa^2 \int_0^\infty d\xi$$

$$\times \ \sigma\varepsilon\xi \ \exp\{-(1-\varepsilon+2\varepsilon\kappa^2)\xi + 2\sigma\alpha\sqrt{2\varepsilon\xi} \ F(\kappa)\} \ . \qquad (4.5)$$

We are interested in the asymptotic expression of this integral for $\varepsilon << 1$. The calculation is a little lengthy but straightforward. The result is given by

$$I_e = e \ n_0 \ S_0 \ v_{ph}(1 - 1.064 \ e^{-\alpha^2} \ \varepsilon^{3/2} + \cdots), \qquad (4.6)$$

where

$$'1.064' = \sqrt{2} \ \{\frac{5}{4} + 3\int_0^1 \frac{1}{u^4}[\frac{\pi}{2E(u)}(1-u^2)-1+ \frac{3}{4} u^2]du\}$$
$$= 1.064 \cdots \qquad (4.7)$$

The first term in the bracket of eq.(4.6) is canceled out by the current carried by ions (note that we are making the analysis in the frame moving with the velocity v_{ph}). Accordingly, the net current is given by

$$I = - 1.064 \ e \ n_0 \ S_0 v_{ph} \ e^{-\alpha^2} \ \varepsilon^{3/2} + \cdots \ . \qquad (4.8)$$

§5. Discussions

1. In a previous paper[1], we used a slightly different model and obtained the same form of expression as eq.(4.7) for the net current. The numerical coefficient is, however, a little different: 1.639 in the previous analysis in place of 1.064 here. The physical basis of the present analysis seems more plausible than that of the previous one, so that we prefer the present one. Since, however, the difference between these two results is small, the conclusion on the required power for the current sustaining in Ref.1 is not much altered.

2. One thing we want to emphasize here is the importance of accurate matching of the expression (3.15) with the Maxwellian distribution for $\kappa >> 1$. If we neglect the constant term in eq.(3.16) as done in the previous papers[2,3], we get in place of eq.(4.8)

$$I = - en_0 S_0 v_{ph} \varepsilon^{1/2} h(\alpha) \quad , \tag{5.1}$$

where

$$h(\alpha) = \sqrt{2}\, c_0\, e^{-\alpha^2}(1 + 2\alpha e^{\alpha^2} \int_0^\alpha e^{-t^2}\, dt) \ . \tag{5.2}$$

This result is unreasonable in the sense not only that I is proportional to $\varepsilon^{1/2}$, but also that the net current increases indefinitely as v_{ph} increases. The origin of the absurdity can be easily traced to the behavior of $f_0(\xi,\kappa,\sigma)$ for $\kappa >> 1$. Accordingly, in our analysis the presence of constant term in eq.(3.16) is essential to obtain a physically reasonable result.

A similar modification may be also important for some other problems. An application to the problem discussed by Zakharov and Karpman[2] will be published in a separate paper.

Acknowledgement

The authors are grateful to Dr. R. Sugihara for very useful discussions and critical reading of the manuscript. They are also indebted to Professor K. Matsuura for stimulating discussions.

One of the authors (Y. M.) wishes to dedicate this paper to Professor Taro Kihara of The University of Tokyo in celebration of his sixtieth birthday.

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