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High-Beta Axisymmetric Equilibria with Flow in Reduced Single-Fluid and Two-Fluid Models

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Reduced single-fluid and two-fluid equations for axisymmetric toroidal equilibria of high-beta plasmas with flow are derived by using asymptotic expansions in terms of the inverse aspect ratio. Two different orderings for the flow velocity, comparable to the poloidal Alfvén velocity and comparable to the poloidal sound velocity, are considered. For a poloidal-Alfvénic flow, the two-fluid equilibrium equations with hot ion effects are shown to have a singularity that is shifted by the gyroviscous cancellation from the Alfvén singularity found in single-fluid magnetohydrodynamics (MHD) when the poloidal flow velocity equals the poloidal Alfvén velocity. For a poloidal-sonic flow, a reduced single-fluid model is used to derive a set of equilibrium equations that includes higher-order terms. The singularity at a poloidal flow velocity equal to the poloidal sound velocity is recovered in the higher order equations.

Keywords: magnetohydrodynamics, plasma equilibrium with flow, two-fluid model, finite Larmor radius, tokamak

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1. Introduction

In improved confinement modes of magnetically confined plasmas where high-β is achieved, equilibrium flows play important roles like the suppression of instability and turbulent transport. At the sharp boundary of a well-confined region, the scale lengths characteristic of microscopic effects not included in single-fluid magnetohydrodynamics (MHD) cannot be neglected. Small scale effects on flowing equilibria due to the Hall current have been studied with two-fluid or Hall MHD models [1–4]. However, these models are consistent with kinetic theory only for cold ions. In order to include the hot ion effects that are relevant to fusion plasmas, an extension of the model is necessary. A consistent treatment of hot ions in a two-fluid framework must include the ion gyroviscosity and other finite Larmor radius (FLR) effects. In the fluid formalism of collisionless magnetized plasmas, these effects are incorporated by means of asymptotic expansions in terms of the small parameter $\delta \sim \rho_i/a$, where ρ_i is the ion Larmor radius and a is the macroscopic scale length. With a slow dynamics ordering, $v \sim \delta v_{th}$ where v and v_{th} are the flow and thermal velocities respectively, the ion FLR terms [5,6] are much simplified in the reduced models for large-aspect-ratio, high-β tokamaks [7, 8] after relating δ to the inverse aspect ratio expansion parameter $\varepsilon \equiv a/R_0 \ll 1$, where a and R_0 are the characteristic scale lengths of the minor and major radii respectively.[9, 10].

In this paper, we derive reduced sets of equations for axisymmetric equilibria with flow. We shall study flow velocities in the orders of the poloidal Alfvén and the poloidal sound velocities. These are the characteristic velocities that bring singularities in the equilibrium equations. The poloidal-Alfvénic flow is of interest because the equations for axisymmetric equilibria in single-fluid MHD have a singularity when the poloidal flow velocity is equal to the poloidal Alfvén velocity, the so-called Alfvén singularity [3, 11]. This can be described by the reduced model with the relation $\delta^2 \sim \epsilon$ [9, 10]. The poloidal-sonic flow is of interest because the equilibria show a discontinuity at the point where the poloidal flow velocity crosses the poloidal sound velocity [12, 13]. This can be described by the reduced model with the relation $\delta \sim \epsilon$. While the poloidal-Alfvénic flow analysis follows the standard orderings of reduced MHD for high-β tokamaks, the poloidal-sonic flow analysis does not and higher-order terms must be taken into account. Since the formulation of higher-order equations is involved, here we restrict our analysis of the poloidal-sonic flow to the single-fluid case, planning to extend our present results with the inclusion of two-fluid, hot ion effects in future work. The orderings in this paper provide the simplest models that include ion FLR effects on toroidal equilibria with flow. As such, they should be just considered as convenient working hypotheses that allow our analytic study of such effects.

This paper is organized as follows. In Sec. 2, we introduce the basic steady state equations for two-fluid MHD with hot ion effects, and the orderings for the reduced models. In Sec. 3, we derive the equations for equilibria with flow velocity comparable to the poloidal Alfvén velocity

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in a model with two-fluid and hot ion effects, and discuss the modification of the Alfvén singularity by these effects. In Sec. 4, we derive the asymptotic equations for equilibria with flow velocity comparable to the poloidal sound velocity in a single fluid model, to be extended to a two-fluid model with ion FLR in future work. A summary is given in Sec. 5.

2. Basic Equations

The equations for two-fluid collisionless equilibria to be considered in this work are

$$\nabla \cdot (n\boldsymbol{v}) = 0, \tag{1}$$

$$\nabla \times \boldsymbol{E} = 0, \tag{2}$$

$$m_{\rm i} n\boldsymbol{v} \cdot \nabla \boldsymbol{v} = \boldsymbol{j} \times \boldsymbol{B} - \nabla (p_{\rm i} + p_{\rm e}) - \lambda_{\rm i} \nabla \cdot \Pi_{\rm i}^{\rm gv}, \tag{3}$$

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \frac{\lambda_{\rm e}}{ne} (\boldsymbol{j} \times \boldsymbol{B} - \nabla p_{\rm e}), \tag{4}$$

$$\mu_0 \boldsymbol{j} = \nabla \times \boldsymbol{B}, \tag{5}$$

$$\boldsymbol{v} \cdot \nabla p_{\rm i} + \gamma p_{\rm i} \nabla \cdot \boldsymbol{v} + \lambda_{\rm i} \left(\frac{2}{5} \gamma \nabla \cdot \boldsymbol{q}_{\rm i} \right) = 0, \tag{6}$$

$$\begin{aligned}(\boldsymbol{v} - \lambda_{\rm e} \boldsymbol{j}/ne) \cdot \nabla p_{\rm e} + \gamma p_{\rm e} \nabla \cdot (\boldsymbol{v} - \lambda_{\rm e} \boldsymbol{j}/ne) \\ + \lambda_{\rm e} \left(\frac{2}{5} \gamma \nabla \cdot \boldsymbol{q}_{\rm e} \right) = 0,\end{aligned} \tag{7}$$

where $m_{\rm i}$ is the ion mass, n is the density, $\boldsymbol{v}$ is the ion flow velocity, $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields, $\boldsymbol{j}$ is the current density, $p_{\rm i}$ and $p_{\rm e}$ are the ion and electron pressures, $\Pi_{\rm i}^{\rm gv}$ is the ion gyroviscous tensor, $\boldsymbol{q}_{\rm i}$ and $\boldsymbol{q}_{\rm e}$ are the ion and electron heat fluxes respectively, and $\gamma = 5/3$. The diagonal components of the pressure tensors are assumed to be isotropic. The electron mass $m_{\rm e}$ is neglected because $m_{\rm e} \ll m_{\rm i}$. The electron gyroviscosity is also neglected since $\rho_{\rm e} \ll \rho_{\rm i}$. We have introduced the artificial indices $\lambda_{\rm i}$ and $\lambda_{\rm e}$ that label the two-fluid, non-ideal terms in the ion and electron equations respectively: $(\lambda_{\rm i}, \lambda_{\rm e}) = (0, 0)$ for single-fluid (ideal) MHD, $(0, 1)$ for two-fluid MHD with electron diamagnetic effects but zero ion Larmor radius (Hall MHD) and $(1, 1)$ for two-fluids with finite ion Larmor radius. The divergence of the ion flow velocity is obtained from the projection of Faraday's law (2) along $\boldsymbol{B}$ and the substitution of the generalized Ohm's law (4),

$$\begin{aligned}&B^2(\nabla \cdot \boldsymbol{v}) + \boldsymbol{v} \cdot \nabla B^2 - \boldsymbol{v} \cdot [\boldsymbol{B} \times (\nabla \times \boldsymbol{B})] \\ &- \boldsymbol{B} \cdot \nabla(\boldsymbol{v} \cdot \boldsymbol{B}) - \boldsymbol{B} \cdot \left[\nabla \left(\frac{\lambda_{\rm e}}{ne} \right) \times \nabla p_{\rm e} \right] \\ &- B^2 \left[\boldsymbol{j} \cdot \nabla \left(\frac{\lambda_{\rm e}}{ne} \right) \right] + \boldsymbol{B} \cdot \boldsymbol{j} \left[\boldsymbol{B} \cdot \nabla \left(\frac{\lambda_{\rm e}}{ne} \right) \right] \\ &+ \frac{\lambda_{\rm e}}{ne} \nabla \cdot [(\boldsymbol{j} \times \boldsymbol{B}) \times \boldsymbol{B}] = 0.\end{aligned} \tag{8}$$

Here we shall consider the corresponding toroidal axisymmetric equilibria, where, in cylindrical coordinates (R, φ, Z), the magnetic field $\boldsymbol{B}$ and the current density $\boldsymbol{j}$ can be written as

$$\boldsymbol{B} = \nabla \psi(R, Z) \times \nabla \varphi + I(R, Z) \nabla \varphi, \tag{9}$$

$$\mu_0 \boldsymbol{j} = \nabla I \times \nabla \varphi - \Delta^* \psi \nabla \varphi, \tag{10}$$

where ψ is the poloidal magnetic flux and $\Delta^* \equiv R^2 \nabla \cdot [R^{-2} \nabla]$. The projection of the momentum balance equation (3) along $\nabla \psi$, $\boldsymbol{B}$ and $\boldsymbol{B}_{\rm p}$ yields

$$\begin{aligned}&\mu_0 R^2 \nabla \psi \cdot (m_{\rm i} n\boldsymbol{v} \cdot \nabla \boldsymbol{v} + \lambda_{\rm i} \nabla \cdot \Pi_{\rm i}^{\rm gv}) + |\nabla \psi|^2 \Delta^* \psi \\ &\quad + I \nabla \psi \cdot \nabla I + \mu_0 R^2 \nabla \psi \cdot \nabla (p_{\rm i} + p_{\rm e}) = 0,\end{aligned} \tag{11}$$

$$\boldsymbol{B} \cdot (m_{\rm i} n\boldsymbol{v} \cdot \nabla \boldsymbol{v} + \lambda_{\rm i} \nabla \cdot \Pi_{\rm i}^{\rm gv}) + \{p_{\rm i} + p_{\rm e},\, \psi\} = 0, \tag{12}$$

$$\begin{aligned}&(\nabla \psi \times \nabla \varphi) \cdot (m_{\rm i} n\boldsymbol{v} \cdot \nabla \boldsymbol{v} + \lambda_{\rm i} \nabla \cdot \Pi_{\rm i}^{\rm gv}) \\ &\quad + \{p_{\rm i} + p_{\rm e},\, \psi\} + (I/\mu_0 R^2)\{I,\, \psi\} = 0,\end{aligned} \tag{13}$$

where $\{a,\, b\} \equiv (\nabla a \times \nabla b) \cdot \nabla \varphi$.

The asymptotic expansion is defined in terms of the inverse aspect ratio $\varepsilon \equiv a/R_0 \ll 1$ where a and R_0 are the characteristic scale lengths of the minor and major radii respectively. The following high-β tokamak orderings for compressible reduced MHD are applied,

$$B_{\rm p} \sim \varepsilon B_0, \tag{14}$$

$$p_{\rm i} \sim p_{\rm e} \sim \varepsilon \left(B_0^2/\mu_0 \right), \tag{15}$$

$$|\nabla| \sim 1/a. \tag{16}$$

The variables are expanded as

$$\begin{aligned}\psi &= \psi_1 + \psi_2 + \psi_3 + \ldots, \\ I &= I_0 + I_1 + I_2 + I_3 + \ldots, \\ p_{\rm i} &= p_{\rm i1} + p_{\rm i2} + p_{\rm i3} + \ldots, \\ p_{\rm e} &= p_{\rm e1} + p_{\rm e2} + p_{\rm e3} + \ldots, \\ n &= n_0 + n_1 + \ldots, \\ R &= R_0 + x,\end{aligned}$$

where $I_0 \equiv B_0 R_0$. We assume the slow dynamics ordering,

$$v \sim \delta v_{\rm thi}, \tag{17}$$

$$m_{\rm i} n v^2 \sim ||\Pi_{\rm i}^{\rm gv}|| \sim \delta^2 p_{\rm i,e}, \tag{18}$$

$$q_{\rm i} \sim v p_{\rm i,e} \sim \delta v_{\rm thi} p_{\rm i,e}. \tag{19}$$

The leading order of Eq. (11) yields

$$p_{\rm i1} + p_{\rm e1} + \frac{B_0}{\mu_0 R_0} I_1 = \text{const.} \tag{20}$$

3. Reduced Two-Fluid Equilibria with Poloidal-Alfvénic Flow

Here, we consider the case of poloidal-Alfvénic flow $v \sim V_{\rm Ap} \equiv B_{\rm p}/(\mu_0 m_{\rm i} n)^{1/2}$,

$$m_{\rm i} n v^2 \sim \left\| \Pi_{\rm i}^{\rm gv} \right\| \sim \varepsilon p \sim \varepsilon^2 B_0^2/\mu_0, \tag{21}$$

and we assume

$$v \sim j/ne \sim \nabla p/neB_0. \tag{22}$$

This requires second-order accuracy in the total energy. However, while second-order accuracy is needed in the sum of the pressures plus the magnetic energy, it follows from Eqs. (11)-(13) that the pressures and the magnetic energy by themselves are required only in the first-order. The ion gyroviscous force and heat fluxes are needed only their leading orders [5,6] (see Appendix for the derivation of the ion gyroviscous force),

$$\nabla \cdot \Pi_{\rm i}^{\rm gv} \simeq -\frac{m_{\rm i}}{eB_0}\left(R_0 \nabla\varphi \times \nabla p_{\rm i}\right) \cdot \nabla \boldsymbol{v} - \nabla\left(\chi_v + \chi_q\right), \tag{23}$$

$$\boldsymbol{q}_{\rm i} \simeq \boldsymbol{q}_{{\rm i}\perp} \simeq \frac{5}{2}\frac{1}{eB^2}\boldsymbol{B} \times \left[p_{\rm i} \nabla\left(\frac{p_{\rm i}}{n}\right)\right], \tag{24}$$

$$\boldsymbol{q}_{\rm e} \simeq \boldsymbol{q}_{{\rm e}\perp} \simeq -\frac{5}{2}\frac{1}{eB^2}\boldsymbol{B} \times \left[p_{\rm e} \nabla\left(\frac{p_{\rm e}}{n}\right)\right], \tag{25}$$

where $\boldsymbol{q}_{{\rm i}\perp}$ and $\boldsymbol{q}_{{\rm e}\perp}$ are the ion and electron diamagnetic perpendicular heat fluxes respectively and their divergences are

$$\nabla \cdot \boldsymbol{q}_{\rm i} \simeq \frac{5}{2}\frac{p_{\rm i1}R_0}{eB_0}\{n_0^{-1},\, p_{\rm i1}\}, \tag{26}$$

$$\nabla \cdot \boldsymbol{q}_{\rm e} \simeq -\frac{5}{2}\frac{p_{\rm e1}R_0}{eB_0}\{n_0^{-1},\, p_{\rm e1}\}. \tag{27}$$

From Faraday's law (2), we obtain

$$\boldsymbol{E} \equiv -\nabla\Phi, \tag{28}$$

and expand Φ as

$$\Phi = \Phi_1 + \ldots$$

The generalized Ohm's law (4) is rewritten as

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \frac{\lambda_{\rm e}}{ne}\left(\nabla p_{\rm i} + m_{\rm i} n \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \lambda_{\rm i} \nabla \cdot \Pi_{\rm i}^{\rm gv}\right). \tag{29}$$

The ion flow velocity $\boldsymbol{v}$ is defined from the leading order of Eq. (29) as

$$\boldsymbol{v} \equiv \boldsymbol{v}_{\rm E} + \lambda_{\rm e}\boldsymbol{v}_{\rm di} + v_{\parallel} R_0 \nabla\varphi, \tag{30}$$

$$\boldsymbol{v}_{\rm E} \simeq -B_0^{-1}\nabla\Phi_1 \times \left(R_0\nabla\varphi\right), \tag{31}$$

$$\boldsymbol{v}_{\rm di} \simeq -\frac{1}{eB_0 n_0}\nabla p_{\rm i1} \times \left(R_0 \nabla\varphi\right), \tag{32}$$

and its divergence is

$$\nabla \cdot \boldsymbol{v} \simeq \lambda_{\rm e} \nabla \cdot \boldsymbol{v}_{\rm di} \simeq -\frac{\lambda_{\rm e} R_0}{eB_0}\{n_0^{-1},\, p_{\rm i1}\}. \tag{33}$$

The leading order of the equation for continuity (1) is

$$-(R_0/B_0)\{n_0,\, \Phi_1\} = 0, \tag{34}$$

which yields

$$n_0 = n_0(\Phi_1). \tag{35}$$

Substituting Eqs. (26), (33) and (35) to Eq. (6), we obtain the leading order ion pressure equation

$$\frac{R_0}{B_0}\left[1 + (\lambda_{\rm e} - \lambda_{\rm i})\frac{n_0'(\Phi_1)\gamma p_{\rm i1}}{en_0^2}\right]\{p_{\rm i1},\, \Phi_1\} = 0, \tag{36}$$

which yields

$$p_{\rm i1} = p_{\rm i1}(\Phi_1). \tag{37}$$

From Eqs. (10) and (20), we get

$$\mu_0 \boldsymbol{j} \simeq -\nabla\left[\frac{\mu_0 R_0}{B_0}(p_{\rm i1} + p_{\rm e1})\right] \times \nabla\varphi - \Delta^*\psi\nabla\varphi. \tag{38}$$

Then, substituting Eqs. (27) and (38), the electron pressure equation (7) gives

$$p_{\rm e1} = p_{\rm e1}(\Phi_1), \tag{39}$$

The leading order of the $\boldsymbol{B}$-component of generalized Ohm's law (4) is

$$-\{\Phi_1,\, \psi_1\} + \frac{\lambda_{\rm e}}{n_0 e}\{p_{\rm e1},\, \psi_1\} = 0. \tag{40}$$

Substituting Eq. (39) into Eq. (40), we obtain

$$-\left[1 - \frac{\lambda_{\rm e} p_{\rm e}'(\Phi_1)}{en_0}\right]\{\Phi_1,\, \psi_1\} = 0, \tag{41}$$

which yields

$$\Phi_1 = \Phi_1(\psi_1). \tag{42}$$

Thus, we get

$$n_0 = n_0(\psi_1), \tag{43}$$

$$p_{\rm i1} = p_{\rm i1}(\psi_1), \tag{44}$$

$$p_{\rm e1} = p_{\rm e1}(\psi_1), \tag{45}$$

$$I_1 = I_1(\psi_1). \tag{46}$$

The convective derivative is written as

$$\begin{aligned} &\boldsymbol{v} \cdot \nabla \boldsymbol{v} \\ &\simeq \frac{1}{B_0^2}\left(\Phi_1' + \frac{\lambda_{\rm e} p_{\rm i1}'}{en_0}\right)^2 \left[\nabla\left(\frac{|\nabla\psi_1|^2}{2}\right) - \Delta_2\psi_1\nabla\psi_1\right] \\ &\quad - \frac{R_0^2}{B_0}\left(\Phi_1' + \frac{\lambda_{\rm e} p_{\rm i1}'}{en_0}\right)\{v_{\parallel},\, \psi_1\}\nabla\varphi, \end{aligned} \tag{47}$$

where

$$\Delta_2 \equiv \left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial Z^2}\right).$$

The projections of the gyroviscous force (23) along $\nabla\psi$, $\boldsymbol{B}$ and $\nabla\psi\times\nabla\varphi$ are

$$\begin{aligned}&\nabla\psi\cdot(\nabla\cdot\Pi_{\rm i}^{\rm gv})\\&\simeq-\frac{m_{\rm i}p_{\rm i1}'}{eB_0^2}\left(\Phi_1'+\frac{\lambda_{\rm e}p_{\rm i1}'}{en_0}\right)\\&\quad\times\left[\nabla\psi_1\cdot\nabla\left(\frac{|\nabla\psi_1|^2}{2}\right)-\Delta_2\psi_1|\nabla\psi_1|^2\right]\\&\quad-\nabla\psi_1\cdot\nabla(\chi_v+\chi_q),\end{aligned}\tag{48}$$

$$\boldsymbol{B}\cdot(\nabla\cdot\Pi_{\rm i}^{\rm gv})\simeq m_{\rm i}n_0R_0(p_{\rm i1}'/en_0)\{v_\parallel,\,\psi_1\},\tag{49}$$

$$\begin{aligned}&(\nabla\psi\times\nabla\varphi)\cdot(\nabla\cdot\Pi_{\rm i}^{\rm gv})\\&\simeq\left\{-\frac{m_{\rm i}p_{\rm i1}'}{eB_0^2}\left(\Phi_1'+\frac{\lambda_{\rm e}p_{\rm i1}'}{en_0}\right)\frac{|\nabla\psi_1|^2}{2},\,\psi_1\right\}\\&\quad+\{\chi_v+\chi_q,\,\psi_1\}.\end{aligned}\tag{50}$$

The components of the sum of the convective derivative and the gyroviscous force are

$$\begin{aligned}&\nabla\psi\cdot(m_{\rm i}n\boldsymbol{v}\cdot\nabla\boldsymbol{v}+\lambda_{\rm i}\nabla\cdot\Pi_{\rm i}^{\rm gv})\\&\simeq-\frac{m_{\rm i}n_0}{B_0^2}\left[\Phi_1'+(\lambda_{\rm e}-\lambda_{\rm i})\frac{p_{\rm i1}'}{en_0}\right]\left(\Phi_1'+\frac{\lambda_{\rm e}p_{\rm i1}'}{en_0}\right)\\&\quad\times\left[|\nabla\psi_1|^2\Delta_2\psi_1-\nabla\psi_1\cdot\nabla\left(\frac{|\nabla\psi_1|^2}{2}\right)\right]\\&\quad-\lambda_{\rm i}\nabla\psi_1\cdot\nabla(\chi_v+\chi_q),\end{aligned}\tag{51}$$

$$\begin{aligned}&\boldsymbol{B}\cdot(m_{\rm i}n\boldsymbol{v}\cdot\nabla\boldsymbol{v}+\lambda_{\rm i}\nabla\cdot\Pi_{\rm i}^{\rm gv})\\&\simeq-m_{\rm i}n_0R_0\left[\Phi_1'+(\lambda_{\rm e}-\lambda_{\rm i})\frac{p_{\rm i1}'}{en_0}\right]\{v_\parallel,\,\psi_1\},\end{aligned}\tag{52}$$

$$\begin{aligned}&(\nabla\psi\times\nabla\varphi)\cdot(m_{\rm i}n\boldsymbol{v}\cdot\nabla\boldsymbol{v}+\lambda_{\rm i}\nabla\cdot\Pi_{\rm i}^{\rm gv})\\&\simeq\left\{\frac{m_{\rm i}n_0}{B_0^2}\left[\Phi_1'+(\lambda_{\rm e}-\lambda_{\rm i})\frac{p_{\rm i1}'}{en_0}\right]\left(\Phi_1'+\frac{\lambda_{\rm e}p_{\rm i1}'}{en_0}\right)\right.\\&\quad\left.\times\frac{|\nabla\psi_1|^2}{2}-\lambda_{\rm i}(\chi_v+\chi_q),\,\psi_1\right\}.\end{aligned}\tag{53}$$

In the first square brackets of Eqs. (51) - (53), the contribution of ion diamagnetic drift disappears when $(\lambda_{\rm i},\lambda_{\rm e})=(1,1)$. This is due to the finite Larmor effect on the convective terms, known as the gyroviscous cancellation [14]. Substituting Eqs. (51) - (53) into Eqs. (11) - (13), we obtain the first-order equations for momentum balance as

$$\begin{aligned}&|\nabla\psi_1|^2\,\Delta_2\psi_1+2\mu_0R_0x\nabla\psi_1\cdot\nabla(p_{\rm i1}+p_{\rm e1})\\&\quad+I_1\nabla\psi_1\cdot\nabla I_1+\mu_0R_0^2\nabla\psi_1\cdot\nabla\left(p_{\rm i2}+p_{\rm e2}+\frac{B_0}{\mu_0R_0}I_2\right)\\&\quad-\frac{\mu_0R_0^2m_{\rm i}n_0}{B_0^2}\left[\Phi_1'+(\lambda_{\rm e}-\lambda_{\rm i})\frac{p_{\rm i1}'}{en_0}\right]\left(\Phi_1'+\frac{\lambda_{\rm e}p_{\rm i1}'}{en_0}\right)\\&\quad\times\left[|\nabla\psi_1|^2\Delta_2\psi_1-\nabla\psi_1\cdot\nabla\left(\frac{|\nabla\psi_1|^2}{2}\right)\right]\\&\quad-\lambda_{\rm i}\mu_0R_0^2\nabla\psi_1\cdot\nabla(\chi_v+\chi_q)=0,\end{aligned}\tag{54}$$

$$-m_{\rm i}n_0\left[\Phi_1'+(\lambda_{\rm e}-\lambda_{\rm i})\frac{p_{\rm i1}'}{en_0}\right]\{v_\parallel,\psi_1\}=0,\tag{55}$$

$$\begin{aligned}&p_{\rm i2}+p_{\rm e2}+\frac{B_0}{\mu_0R_0}I_2\\&\quad+\frac{m_{\rm i}n_0}{B_0^2}\left[\Phi_1'+\frac{(\lambda_{\rm e}-\lambda_{\rm i})p_{\rm i1}'}{en_0}\right]\left(\Phi_1'+\frac{\lambda_{\rm e}p_{\rm i1}'}{en_0}\right)\frac{|\nabla\psi_1|^2}{2}\\&\quad-\lambda_{\rm i}(\chi_v+\chi_q)\equiv g_*(\psi_1).\end{aligned}\tag{56}$$

Equation (55) yields

$$v_\parallel=v_\parallel(\psi_1).\tag{57}$$

Substituting Eq. (56) to Eq. (54), we get the reduced Grad-Shafranov (GS) equation in the presence of flow, two-fluid and ion FLR effects,

$$\begin{aligned}&\left[1-M_{\rm Ap}\left(M_{\rm Ap}-\lambda_{\rm i}\frac{V_{\rm di}}{V_{\rm Ap}}\right)\right]\Delta_2\psi_1\\&\quad-\frac{|\nabla\psi_1|^2}{2}\left[M_{\rm Ap}\left(M_{\rm Ap}-\lambda_{\rm i}\frac{V_{\rm di}}{V_{\rm Ap}}\right)\right]'\\&\quad=-\mu_0R_0^2\left[\frac{2x}{R_0}(p_{\rm i1}+p_{\rm e1})'+g_*'\right]-\left(\frac{I_1^2}{2}\right)',\end{aligned}\tag{58}$$

where

$$M_{\rm Ap}\equiv\frac{V_{\rm E}+\lambda_{\rm e}V_{\rm di}}{V_{\rm Ap}},\tag{59}$$

is the poloidal Alfvén Mach number and

$$\frac{V_{\rm E}}{V_{\rm Ap}}\equiv-\sqrt{\mu_0m_{\rm i}n_0}\frac{R_0\Phi_1'}{B_0},\tag{60}$$

$$\frac{V_{\rm di}}{V_{\rm Ap}}\equiv-\sqrt{\mu_0m_{\rm i}n_0}\frac{R_0p_{\rm i1}'}{en_0B_0}.\tag{61}$$

Equation (58) has a singularity where the first term of the left-hand side vanishes,

$$1-M_{\rm Ap}\left(M_{\rm Ap}-\lambda_{\rm i}\frac{V_{\rm di}}{V_{\rm Ap}}\right)=0,\tag{62}$$

or

$$V_{\rm Ap}^2-(V_{\rm E}+\lambda_{\rm e}V_{\rm di})[V_{\rm E}+(\lambda_{\rm e}-\lambda_{\rm i})V_{\rm di}]=0.\tag{63}$$

For single fluid MHD, $(\lambda_{\rm i},\,\lambda_{\rm e})=(0,0)$, it is the Alfvén singularity that occurs when the poloidal flow velocity is equal to the poloidal Alfvén velocity,

$$M_{\rm AP}^2=1.\tag{64}$$

For two-fluid MHD without ion FLR, $(\lambda_{\rm i},\,\lambda_{\rm e})=(0,1)$, the condition is the same as Eq. (64) even though the two-fluid effects bring the ion diamagnetic drift into the definition of the poloidal flow. For the two-fluid model with ion FLR, $(\lambda_{\rm i},\,\lambda_{\rm e})=(1,1)$, the singularity is shifted from the poloidal Alfvén velocity,

$$M_{\rm Ap}=\frac{1}{2}\left[V_{\rm di}\big/V_{\rm Ap}\pm\sqrt{4+\left(V_{\rm di}\big/V_{\rm Ap}\right)^2}\right].\tag{65}$$

From Eq. (63), this shift is understood as the effect of the gyroviscous cancellation on flowing equilibria.

It is noted that the present model does not reproduce the resolution of the Alfvén singularity, Eq. (64), by the Hall current as in non-reduced two-fluid models with $(\lambda_i, \lambda_e) = (0, 1)$ [3]. This difference arises because the convective term in the ion momentum balance equation (29) is neglected in the leading order. This convective term involves the second order derivative of the ion stream function and leads to an equilibrium system of two coupled generalized GS equations for the ion flow stream function and ψ, which does not have the Alfvén singularity [1, 3, 4, 15]. In order to describe the singular perturbation due to the Hall current in reduced models, the local region in the vicinity of the Alfvén singularity should be separately analysed by relaxing the ordering $B_p \sim \varepsilon B_0$ and connected to the bulk region described by Eq. (58). Equation (65) specifies the region where the singular perturbation analysis is necessary in the FLR two-fluid model.

4. Reduced Single-Fluid Equilibria with Poloidal-Sonic Flow

This section will deal with single-fluid equilibria, $(\lambda_i, \lambda_e) = (0, 0)$, with the flow velocity in the order of the poloidal sound speed $v \sim C_{sp} \equiv (B_p/B_0)(\gamma p/nm_i)^{1/2}$,

$$m_i n v^2 \sim \varepsilon^2 p \sim \varepsilon^3 \left(B_0^2/\mu_0\right).$$

This requires a third-order accuracy for the total energy like the reduced MHD equations for finite aspect ratio tokamaks [8]. However, like in the previous section, the pressures and the magnetic energy by themselves are required only up to the second order. From Eqs. (6) and (7), the total pressure $p = p_i + p_e$ is given by the adiabatic pressure equation,

$$\boldsymbol{v} \cdot \nabla p + \gamma p \nabla \cdot \boldsymbol{v} = 0. \tag{66}$$

The fast magnetosonic wave is eliminated from the reduced equations for equilibria, while the shear Alfvén and the slow magnetosonic waves are retained, by assuming

$$\nabla \cdot \boldsymbol{v} \sim \varepsilon v/a. \tag{67}$$

From the requirements Eq. (67) and $\boldsymbol{v} \cdot \nabla p \sim \varepsilon^2 (B_0^2/\mu_0)(v/a)$ that is determined from Eq. (66) and satisfied by Eq. (8), the flow velocity $\boldsymbol{v}$ can be written as [8]

$$\begin{aligned}\boldsymbol{v} &\equiv (1 + x/R_0)\, \nabla U \times (\boldsymbol{B}/B) + v_\parallel\, (\boldsymbol{B}/B)\\ &\equiv \boldsymbol{v}_p + v_\varphi R \nabla\varphi,\end{aligned} \tag{68}$$

$$\boldsymbol{v}_p \equiv \left[\frac{v_\parallel}{B}\nabla\psi + \left(1 + \frac{x}{R_0}\right)\frac{I}{B}\nabla U\right] \times \nabla\varphi, \tag{69}$$

$$v_\varphi R \equiv \frac{I v_\parallel}{B} - \left(1 + \frac{x}{R_0}\right)\frac{\nabla\psi \cdot \nabla U}{B}. \tag{70}$$

The convective derivative terms are written as

$$\begin{aligned}&\nabla\left(\frac{v^2}{2}\right)\\ =&\frac{1}{2}\nabla\left\{\left(1 + \frac{x}{R_0}\right)^2\left[|\nabla U|^2 - \left(\frac{\boldsymbol{B}}{B}\cdot\nabla U\right)^2\right] + v_\parallel^2\right\},\end{aligned} \tag{71}$$

$$\begin{aligned}&\boldsymbol{v} \times (\nabla \times \boldsymbol{v})\\ =&\left\{\nabla\left[\frac{I}{B}v_\parallel - \left(1 + \frac{x}{R_0}\right)\frac{\nabla\psi\cdot\nabla U}{B}\right] \times \nabla\varphi\right\}\\ &\cdot\left[\frac{v_\parallel}{B}\nabla\psi + \left(1 + \frac{x}{R_0}\right)\frac{I}{B}\nabla U\right]\nabla\varphi\\ &+\frac{1}{R^2}\left\{\frac{v_\parallel}{B}\Delta^*\psi + \left(1 + \frac{x}{R_0}\right)\frac{I}{B}\Delta^* U\right.\\ &\left.+\nabla\left(\frac{v_\parallel}{B}\right)\cdot\nabla\psi + \nabla\left[\left(1 + \frac{x}{R_0}\right)\frac{I}{B}\right]\cdot\nabla U\right\}\\ &\times\left[\frac{v_\parallel}{B}\nabla\psi + \left(1 + \frac{x}{R_0}\right)\frac{I}{B}\nabla U\right]\\ &+\frac{1}{2R^2}\nabla\left[\frac{I}{B}v_\parallel - \left(1 + \frac{x}{R_0}\right)\frac{\nabla\psi\cdot\nabla U}{B}\right]^2.\end{aligned} \tag{72}$$

The function U is expanded as

$$U = U_1 + U_2 + \ldots$$

In the leading order, the flow velocity $\boldsymbol{v}$ is written in the standard representation for incompressible flow

$$\boldsymbol{v} = R_0(\nabla U_1 \times \nabla\varphi + v_\parallel \nabla\varphi). \tag{73}$$

The leading order of the φ-component of Ohm's law (4) yields

$$U_1 = U_1(\psi_1), \tag{74}$$

and its next order is

$$R_0(\{U_2, \psi_1\} + \{U_1, \psi_2\}) = 0, \tag{75}$$

which yields

$$U_2 - U_1'\psi_2 \equiv U_{2*}(\psi_1), \tag{76}$$

where the prime denotes the derivative with respect to ψ_1. The lowest order of $\nabla \cdot \boldsymbol{v}$ is obtained from the projection of Faraday's law (2) along $\boldsymbol{B}$ as

$$\nabla \cdot \boldsymbol{v} \simeq \left\{\frac{v_\parallel}{B_0} + 2xU_1', \psi_1\right\}. \tag{77}$$

The second term in the bracket of Eq. (77) represents the compressibility of the perpendicular ($\boldsymbol{E} \times \boldsymbol{B}$) flow $\boldsymbol{v}_\perp$ due to toroidicity, that may give rise to the geodesic acoustic mode (GAM) [16],

$$\nabla \cdot \boldsymbol{v}_\perp = B^{-4}(\boldsymbol{E} \times \boldsymbol{B}) \cdot \nabla B^2. \tag{78}$$

The leading order of the pressure equation (66) is

$$R_0 U_1'\{p_1, \psi_1\} = 0, \tag{79}$$

which yields

$$p_1 = p_1(\psi_1), \tag{80}$$

and the next order is

$$R_0(\{p_2, U_1\} + \{p_1, U_2\}) = -\gamma p_1(\nabla \cdot \boldsymbol{v}). \tag{81}$$

Substituting Eq. (77) into Eq. (81), one obtains the equation for the second order pressure,

$$p_2 - p_1'\psi_2 + \gamma p_1 \left(\frac{v_\parallel}{B_0 R_0 U_1'} + \frac{2x}{R_0} \right) \equiv p_{2*}(\psi_1). \tag{82}$$

Analogously, the continuity equation (1) gives the equations for the zeroth- and first-order density,

$$n_0 = n_0(\psi_1), \tag{83}$$

$$n_1 - n_0'\psi_2 + \frac{n_0 v_\parallel}{B_0 R_0 U_1'} + \frac{2x}{R_0} n_0 \equiv n_{1*}(\psi_1). \tag{84}$$

The first order of Eq. (12) is

$$\{m_\mathrm{i} n_0 B_0 R_0 v_\parallel,\, U_1\} + \{p_2,\, \psi_1\} + \{p_1,\, \psi_2\} = 0, \tag{85}$$

which yields the equation for $v_\parallel$,

$$B_0 R_0 m_\mathrm{i} n_0 U_1' v_\parallel + p_2 - p_1'\psi_2 \equiv p_{3*}(\psi_1), \tag{86}$$

which is the Bernoulli law in the present system. Equations (82), (84) and (86) indicate the coupling of $v_\parallel$, p_2 and n_1 due to the slow magnetosonic wave which is lost in the cold ($p_1 \to 0$) or incompressible ($\gamma \to \infty$) limits, and yield

$$v_\parallel = -\frac{(2x/R_0)\gamma p_1 - (p_{2*} - p_{3*})}{(\beta_1 - M_\mathrm{Ap}^2)(B_0^2/\mu_0)} M_\mathrm{Ap} v_\mathrm{A}, \tag{87}$$

$$p_2 = p_1'\psi_2 + \left(\frac{2x}{R_0}\right) \frac{M_\mathrm{Ap}^2 \gamma p_1}{\beta_1 - M_\mathrm{Ap}^2} - \frac{M_\mathrm{Ap}^2 p_{2*} - \beta_1 p_{3*}}{\beta_1 - M_\mathrm{Ap}^2}, \tag{88}$$

$$n_1 = n_0'\psi_2 + n_{1*} + \left(\frac{2x}{R_0}\right) \frac{M_\mathrm{Ap}^2 n_0}{\beta_1 - M_\mathrm{Ap}^2} - \frac{(p_{2*} - p_{3*}) n_0}{(\beta_1 - M_\mathrm{Ap}^2)(B_0^2/\mu_0)}, \tag{89}$$

where $\beta_1 \equiv \gamma p_1/(B_0^2/\mu_0)$, $v_\mathrm{A} \equiv B_0/\sqrt{\mu_0 n_0 m_\mathrm{i}}$ is the Alfvén velocity and $M_\mathrm{Ap}(\psi_1) \equiv v_\mathrm{p}/V_\mathrm{Ap} \equiv (\mu_0 m_\mathrm{i} n_0)^{1/2} R_0 U_1'$ is the poloidal Alfvén Mach number. The singularity appears when $\beta_1 = M_\mathrm{Ap}^2$, i.e. when the poloidal flow velocity equals to the poloidal sound velocity. The first and second orders of Eq. (11) are

$$|\nabla\psi_1|^2 \Delta_2\psi_1 + 2\mu_0 R_0 x \nabla\psi_1 \cdot \nabla p_1 + I_1 \nabla\psi_1 \cdot \nabla I_1 + \mu_0 R_0^2 \nabla\psi_1 \cdot \nabla p_2 + B_0 R_0 \nabla\psi_1 \cdot \nabla I_2 = 0, \tag{90}$$

and

$$\begin{aligned} &|\nabla\psi_1|^2 \left(\Delta_2\psi_2 - \frac{1}{R}\frac{\partial\psi_1}{\partial R} \right) + 2\,(\nabla\psi_1 \cdot \nabla\psi_2)\,\Delta_2\psi_1 \\ &+ \mu_0 x^2 \nabla\psi_1 \cdot \nabla p_1 + \nabla\psi_2 \cdot \nabla \left(I_1^2/2 \right) \\ &+ \nabla\psi_2 \cdot \nabla \left(\mu_0 R_0^2 p_2 + B_0 R_0 I_2 \right) \\ &+ 2\mu_0 R_0 x\, (\nabla\psi_2 \cdot \nabla p_1 + \nabla\psi_1 \cdot \nabla p_2) \\ &+ \nabla\psi_1 \cdot \nabla \left(\mu_0 R_0^2 p_3 + R_0 B_0 I_3 + I_1 I_2 \right) \\ &- \mu_0 m_\mathrm{i} n_0 R_0^2\, (\nabla\psi_1 \cdot \nabla U_1)\, \Delta_2 U_1 \\ &+ \mu_0 m_\mathrm{i} n_0 R_0^2 \nabla\psi_1 \cdot \nabla \left(|\nabla U_1|^2/2 \right) = 0. \end{aligned} \tag{91}$$

The first order of Eq. (13) yields

$$p_2 + \frac{B_0}{\mu_0 R_0} I_2 \equiv g_*(\psi_1), \tag{92}$$

and the second order is

$$\begin{aligned} &\left\{ p_3 + \frac{B_0}{\mu_0 R_0} I_3,\, \psi_1 \right\} + \frac{2x}{R_0} \{p_2 - p_1'\psi_2,\, \psi_1\} \\ &+ \frac{I_1}{\mu_0 R_0^2} \{I_2 - I_1'\psi_2,\, \psi_1\} - \{g_*'\psi_2,\, \psi_1\} \\ &+ m_\mathrm{i} n_0 U_1'^2 \left\{ \frac{|\nabla\psi_1|^2}{2},\, \psi_1 \right\} = 0. \end{aligned} \tag{93}$$

Substituting Eq. (88) into Eq. (93), we get

$$\begin{aligned} &p_3 + \frac{B_0 I_3}{\mu_0 R_0} + \frac{I_1}{\mu_0 R_0^2} (I_2 - I_1'\psi_2) + m_\mathrm{i} n_0 U_1'^2 \frac{|\nabla\psi_1|^2}{2} \\ &+ \left(\frac{x}{R_0}\right)^2 \frac{2M_\mathrm{Ap}^2 \gamma p_1}{\beta_1 - M_\mathrm{Ap}^2} - g_*'\psi_2 \equiv E_*(\psi_1). \end{aligned} \tag{94}$$

Substituting Eqs. (92) and (94) into Eqs. (90) and (91), we obtain the expanded GS equation in the presence of poloidal-sonic flow,

$$\Delta_2\psi_1 = -\mu_0 R_0^2 \left(\frac{2x}{R_0} p_1' + g_*' \right) - \left(\frac{I_1^2}{2} \right)', \tag{95}$$

$$\begin{aligned} &\Delta_2\psi_2 + \left[\mu_0 R_0^2 \left(\frac{2x}{R_0} p_1'' + g_*'' \right) + \left(\frac{I_1^2}{2} \right)'' \right] \psi_2 \\ &= \frac{1}{R}\frac{\partial\psi_1}{\partial R} + M_\mathrm{Ap}^2 \Delta_2\psi_1 + \frac{|\nabla\psi_1|^2}{2} \left(M_\mathrm{Ap}^2 \right)' \\ &\quad - \mu_0 R_0^2 \left[E_*' + \left(\frac{x}{R_0}\right)^2 p_1' + \left(\frac{x}{R_0}\right)^2 \left(\frac{2M_\mathrm{Ap}^2 \gamma p_1}{\beta_1 - M_\mathrm{Ap}^2} \right)' \right. \\ &\quad \left. - \frac{2x}{R_0} \left(\frac{M_\mathrm{Ap}^2 p_{2*} - \beta_1 p_{3*}}{\beta_1 - M_\mathrm{Ap}^2} \right)' \right]. \end{aligned} \tag{96}$$

The equation for ψ_1 (95) is same as for the static case while the equation for ψ_2 (96) is modified by the flow and has the singularity. Comparing with the analysis of the transonic flow for low-β tokamaks [13, 17], the singularity at the poloidal flow velocity equal to poloidal sound velocity in the density and pressure and its dependence on toroidicity have been reproduced as higher-order effects and the singularity in the higher order magnetic structure has been found in the present study. However, in order to reproduce the radial discontinuity of the density and the pressure found in the low-β analysis [13], a local analysis in the vicinity of the singularity where $\beta_1 - M_{\rm Ap}^2 \sim \varepsilon M_{\rm Ap}^2$ may be necessary. Finally, we note that the hyperbolic region between the cusp velocity and the poloidal velocity of the slow magnetosonic wave pointed out in Ref. [18] degenerates to the singularity in our present ordering, because the difference between its upper and lower bounds becomes of higher order.

5. Summary

We have derived the equations for high-β axisymmetric equilibria with flow comparable to the poloidal Alfvén velocity in the reduced two-fluid model with FLR and flow comparable to the poloidal sound velocity in the single-fluid model, by using asymptotic expansions in terms of the inverse aspect ratio. We have shown that the Alfvén singularity is shifted by the gyroviscous cancellation. The singularity at the poloidal flow velocity equal to the poloidal sound velocity in the density and pressure and its dependence on toroidicity have been reproduced by our higher-order terms and the singularity in the higher-order magnetic structure has been found. The reduced single-fluid equations for equilibria with poloidal-sonic flow include higher-order quantities and hence can describe finite-aspect-ratio tokamak equilibria. The resulting equations can be easily solved numerically to yield flowing equilibria without singularity and their solutions can be used as initial states or for comparison with saturated states of reduced model nonlinear simulations.

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Appendix: Ion Gyroviscous Force

The gyroviscous force is given by [6]

$$\nabla\cdot\Pi_{\rm i}^{\rm gv} = \nabla\cdot\left(\sum_{N=1}^{5}\Pi_{\rm i}^{{\rm gv}N}\right) \tag{A.1}$$

In the ordering of Sec. 2, the leading order terms of Eq. (A.1) are

$$\begin{aligned}
&\nabla\cdot\Pi_{\rm i}^{\rm gv1}\\
&\simeq -m_{\rm i}n\boldsymbol{v}_{*{\rm i}}\cdot\nabla\boldsymbol{v} - \nabla\chi_v - \nabla\times\left[\frac{m_{\rm i}p_{\rm i}}{2eB^2}(\nabla\cdot\boldsymbol{v})\boldsymbol{B}\right]\\
&\simeq -m_{\rm i}n\boldsymbol{v}_{*{\rm i}}\cdot\nabla\boldsymbol{v} - \nabla\chi_v\\
&\quad + \frac{m_{\rm i}R_0^2}{4e^2B_0^2}\nabla\varphi\times\nabla\left\{\nabla\varphi\cdot\left[\nabla p_{\rm i}^2\times\nabla\left(\frac{1}{n}\right)\right]\right\},
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
&\nabla\cdot\Pi_{\rm i}^{\rm gv2}\\
&\simeq -\nabla\chi_q + \frac{m_{\rm i}R_0}{4eB_0}\nabla\varphi\times\nabla\left(\frac{4}{5}\nabla\cdot\boldsymbol{q}_{{\rm i}\perp}\right)\\
&\simeq -\nabla\chi_q\\
&\quad - \frac{m_{\rm i}R_0^2}{4e^2B_0^2}\nabla\varphi\times\nabla\left\{\nabla\varphi\cdot\left[\nabla p_{\rm i}^2\times\nabla\left(\frac{1}{n}\right)\right]\right\},
\end{aligned} \tag{A.3}$$

$$\nabla\cdot\Pi_{\rm i}^{\rm gv3} \simeq \nabla\cdot\Pi_{\rm i}^{\rm gv4} \simeq \nabla\cdot\Pi_{\rm i}^{\rm gv5} \simeq 0, \tag{A.4}$$

where

$$\begin{aligned}
\boldsymbol{v}_{*{\rm i}} &\equiv -\frac{1}{en}\nabla\times\left(\frac{p_{\rm i}}{B^2}\boldsymbol{B}\right)\\
&\simeq \frac{R_0}{enB_0}\nabla\varphi\times\nabla p_{\rm i},
\end{aligned} \tag{A.5}$$

$$\chi_v \equiv \frac{m_{\rm i}p_{\rm i}}{2eB^2}\boldsymbol{B}\cdot(\nabla\times\boldsymbol{v}), \tag{A.6}$$

$$\chi_q \equiv \frac{m_{\rm i}}{5eB^2}\boldsymbol{B}\cdot(\nabla\times\boldsymbol{q}_{{\rm i}\perp}). \tag{A.7}$$

Then we obtain the representation of the ion gyroviscous force, Eq. (23).

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