§48. Development of Meshless Approach by Implicit Surface

Kamitani, A., Takayama, T. (Yamagata Univ.),
Nakata, S. (Ritsumeikan Univ.),
Ikuno, S., Hanawa, T. (Tokyo Univ. Tech.),
Itoh, T. (Seikei Univ.),
Nakamura, H., Tamura, Y.

i) Introduction

Mukherjee *et al.*[1] proposed the boundary node method (BNM) for solving the boundary-value problem of a partial differential equation. Since the BNM is one of meshless approaches, elements of a geometrical structure are no longer necessary and, hence, the preparation of data is considerably simplified. However, the BNM has been plagued by its inherent difficulty: the boundary must be divided into a set of cells to evaluate contour integrals. Hence, a concept of elements is partly included into the BNM.

The purpose of the present study is to formulate the boundary node method without using any integration cells and to investigate its performance by comparing with the boundary element method (BEM).

ii) Boundary Node Method without Cells

As a potential problem, we consider a two-dimensional (2-D) Laplace problem in the region Ω bounded by $\partial\Omega$. As is well known, a 2-D Laplace equation is equivalent to the following boundary integral equation:

$$\oint_{\partial\Omega}\left\{q(\boldsymbol{x}(s))w^{*}-[u(\boldsymbol{x}(s))-u(\boldsymbol{y})]\frac{\partial w^{*}}{\partial n}\right\}ds=0\,, \qquad (1)$$

where $w^{*}(\boldsymbol{x},\boldsymbol{y})=-\log|\boldsymbol{x}-\boldsymbol{y}|/(2\pi)$, $q\equiv\partial u/\partial n$ and s is an arclength along $\partial\Omega$. Two numerical techniques are indispensable for discretizing eq. (1): the one is the method for evaluating a contour integral along $\partial\Omega$ and the other is the approximation for the distribution of u and q on $\partial\Omega$.

First of all, let us explain the numerical method for evaluating a contour integral. In the conventional BNM, integration cells are employed for this evaluation. In contrast, the integral is directly calculated by use of the vector equation of $\partial\Omega$ in the present study. To this end, the implicit-function representation $f(\boldsymbol{x})=0$ [2] is first determined for the curve that passes through all nodes, $\boldsymbol{x}_1,\boldsymbol{x}_2,\cdots,\boldsymbol{x}_N$, on $\partial\Omega$. Subsequently, the vector equation $\boldsymbol{x}=\boldsymbol{x}(s)$ of $\partial\Omega$ is determined by numerically solving the following ordinary differential equation:

$$\frac{d\boldsymbol{x}}{ds}=\boldsymbol{R}\left(\frac{\pi}{2}\right)\cdot\frac{\nabla f}{|\nabla f|}\,, \qquad (2)$$

where $\boldsymbol{R}(\theta)$ denotes a tensor representing a rotation through an angle θ. By using the resulting vector equation of $\partial\Omega$, contour integrals can be evaluated without cells.

Next, for the purpose of approximating $u(s)$ and $q(s)$ on $\partial\Omega$, the periodic shape functions are introduced by means of the moving least-squares approximation and, subsequently, both $u(s)$ and $q(s)$ are assumed to be contained in the functional space spanned by the shape functions.

By means of the above two techniques, eq. (1) and its associated boundary condition can be discretized to a linear system. Throughout the present study, the above method is called a Boundary Node Method Without Cells (BNMWC).

iii) Numerical Results

As an example problem, we adopt the 2-D Laplace problem over $\Omega\equiv\{(x,y):(x/2)^2+y^2<1\}$ with the Dirichlet condition: $u=\cosh x\sin y+\cos x\sinh y$ on $\partial\Omega$. In order to compare the accuracy of the BNMWC with that of the BEM, the relative errors between the numerical and the analytic solutions are calculated as a function of N and are depicted in Fig. 1. We see from this figure that, for both methods, the relative errors are almost proportional to $N^{-\alpha}$ and that the power indices α's satisfy $\alpha\approx1.2$, $\alpha\approx2.0$ and $\alpha\approx3.4$ for the linear BEM, the quadratic BEM and the BNMWC, respectively. This result means that the accuracy of the BNMWC is much higher than that of the BEM.

Consequently, we might conclude that the BNMWC is a powerful method for solving a potential problem.

1) Mukherjee, Y., and Mukherjee, S.: Int. J. Numer. Meth. Eng. 40 (1997) 797.
2) Ohtake, Y. *et al.*: ACM Trans. Graphics 22 (2003) 463.

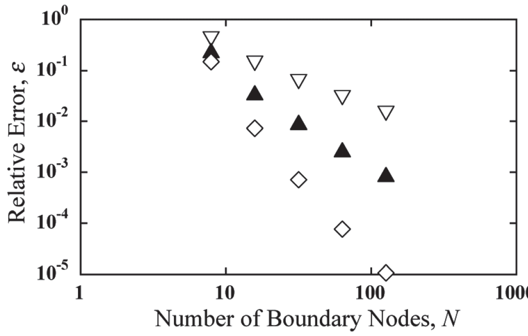

Fig. 1. Dependence of the relative error on the number N of the nodes. Here, ▽ : the linear BEM (m=2), ▲ : the quadratic BEM (m=3), ◇ : the BNMWC (m=2).