J. Plasma Fusion Res. SERIES, Vol. 6 (2004) 271–274

Spectrum of the Velocity Distribution Function in the Slab Ion Temperature Gradient Driven Turbulence

SUGAMA Hideo and WATANABE Tomohiko

National Institute for Fusion Science, Graduate University for Advanced Studies, Toki 509-5292, Japan

(Received: 10 December 2003 / Accepted: 12 February 2004)

Abstract

A spectral analysis of the velocity distribution function in the slab ion temperature gradient (ITG) driven turbulence is done by using high-resolution Eulerian kinetic simulation results. It is clarified how the entropy variable associated with the fine-scale structure of the distribution function is produced by the turbulent heat transport in the presence of the temperature gradient, transferred from macro to microscales in the velocity space through phase-mixing processes, and dissipated by collisions. An interesting analogy between the entropy spectrum in the ITG turbulence and the spectrum of a passive scalar quantity in a turbulent fluid is pointed out. The entropy spectral function is analytically derived and confirmed by the simulation result. It is shown that the entropy spectrum obeys a power law in the range that is free from instability sources and collisional dissipation.

Keywords:

slab ITG turbulence, entropy spectrum, gyrokinetic equation, anomalous transport

1. Introduction

Plasma turbulence and resultant anomalous transport have long been a key issue in the magnetic fusion research [1]. Complexity of the issue partly lies in the fact that, for qualitative study of turbulence in high-temperature plasmas, we should take account of kinetic characteristics and accordingly treat not only real-space but also velocity-space structures of the particle distribution function. Because of large computer memory and time required for calculation of the distribution function in such turbulent kinetic systems, detailed investigations of its velocity-space structures in turbulent states have not much been made so far compared with those on the real-space structures of fluid variables. In the present work, combining high-resolution Eulerian kinetic simulation [2,3] with theoretical analysis, we elucidate the velocity-space spectral structures in the slab ion temperature gradient (ITG) driven turbulence [4,5] which causes anomalous transport in high-temperature plasmas. This typical example of plasma turbulence involves a turbulent $\boldsymbol{E}\times\boldsymbol{B}$ flow and anomalous heat transport in the real space as well as phase-mixing and collisional dissipation processes in the velocity space.

This paper is organized as follows. A basic kinetic equation for the slab ITG turbulence and geometrical conditions are described in Sec. **2**. The spectral analysis of the distribution function is presented in Sec. **3**. There, analytically derived entropy spectral functions are compared with results from numerical simulation of the slab ITG turbulence. Finally, conclusions are given in Sec. **4**.

2. Basic kinetic equation

As in the previous work on the collisionless ITG turbulence [4], we consider a periodic two-dimensional slab configuration with translational symmetry in the z-direction, where the uniform magnetic field is set in the y-z plane such that $\boldsymbol{B} = B(\hat{z} + \theta\hat{y})$ with $\theta \ll 1$. Assuming the fluctuation part of the ion distribution function to be give by $\delta f_{\boldsymbol{k}}(v_\parallel, v_\perp) = \tilde{f}_{\boldsymbol{k}}(v_\parallel) F_M(v_\perp)$, with the Maxwellian velocity distribution F_M, neglecting the parallel nonlinear term and taking a $v_\perp$-integral of the ion gyrokinetic equation yield the following equation in the wave number space $\boldsymbol{k} = (k_x, k_y)$,

$$\partial_t \tilde{f}_{\boldsymbol{k}} + i\Theta v_\parallel k_y \tilde{f}_{\boldsymbol{k}} + \sum_{\boldsymbol{k}=\boldsymbol{k}'+\boldsymbol{k}''} (k'_y k''_x - k'_x k''_y)\Psi_{\boldsymbol{k}'}\tilde{f}_{\boldsymbol{k}''}$$
$$= -ik_y \Psi_{\boldsymbol{k}}\left[1 + (v_\parallel^2 - 1 - k^2)\eta_i/2 + \Theta v_\parallel\right] F_M(v_\parallel) + C_i(\tilde{f}_{\boldsymbol{k}}), \quad (1)$$

where we also have assumed constant density and temperature gradients of the background ions in the x-direction with much larger scale-lengths [$L_n \equiv -\mathrm{d}(\ln n)/\mathrm{d}x$ and $L_T \equiv -\mathrm{d}(\ln T_i)/\mathrm{d}x$] than the fluctuation wave lengths. The electric potential $\phi_{\boldsymbol{k}}$ is related to $\Psi_{\boldsymbol{k}}$ by $\Psi_{\boldsymbol{k}} = \mathrm{e}^{-k^2/2}\phi_{\boldsymbol{k}}$ with $k^2 = k_x^2 + k_y^2$. In addition to eq. (1), the electron Boltzmann relation and the quasineutrality condition, which are not written here, are used to give a closed system of governing equations for the slab ITG turbulence [2,4,5]. In eq. (1), we have used the following normalization; $x = x'/\rho_i$, $y = y'/\rho_i$, $v_\parallel = v'_\parallel/v_{ti}$, $t = t'v_{ti}/L_n$, $\tilde{f} = \tilde{f}'L_n v_{ti}/\rho_i n_0$, and $\phi = e\phi' L_n/T_i\rho_i$, where v_{ti}, ρ_i ($= v_{ti}/\Omega_i$), Ω_i, n_0, e, and T_i are the ion thermal velocity, the ion thermal gyro-

Corresponding author's e-mail: sugama@nifs.ac.jp

radius, the ion cyclotron frequency, the background plasma density, the elementary charge, and the background ion temperature ($T_i = m_i v_{ti}^2$; m_i means the ion mass), respectively. Prime means a dimensional quantity. Θ is defined as $\Theta = \theta L_n/\rho_i$. η_i is given by $\eta_i = L_n/L_T$.

The parallel advection term on the left hand side of eq. (1) contributes to generation of fine-scale fluctuations of $\tilde{f}_k$ in the velocity space, that is, the phase mixing. The instability drive is contained in the first group of terms on the right-hand side of eq. (1). The last term on the right-hand side denotes the ion-ion collision term for which we employ the Lenard-Bernstein model collision operator, $C_i(\tilde{f}_k) = \nu\partial_{v_\parallel}[\partial_{v_\parallel} + v_\parallel]\tilde{f}_k(v_\parallel)$, with the collision frequency ν normalized by v_{ti}/L_n.

3. Spectral analysis

In order to investigate the velocity-space structure of the distribution function, we expand $\tilde{f}_k$ as

$$\tilde{f}_{\boldsymbol{k}}(v_\parallel) = \sum_{n=0}^{\infty} \tilde{f}_{\boldsymbol{k},n} H_n(v_\parallel) F_M(v_\parallel), \tag{2}$$

where $\mathrm{H}_n(v_\parallel)$ is the Hermite polynomial of order n. In terms of the coefficient $\hat{f}_{\boldsymbol{k},n}$ in the Hermite-polynomial expansion, eq. (1) is rewritten by

$$\begin{aligned} &\frac{\partial \hat{f}_{\boldsymbol{k},n}}{\partial t} + ik_y\Theta\left[\hat{f}_{\boldsymbol{k},n-1} + (n+1)\hat{f}_{\boldsymbol{k},n-1}\right] \\ &+ \sum_{\boldsymbol{k}=\boldsymbol{k}'+\boldsymbol{k}''} (k_y' k_x'' - k_x' k_y'')\Psi_{\boldsymbol{k}'}\hat{f}_{\boldsymbol{k}'',n} \\ &= -ik_y\Psi_{\boldsymbol{k}}\left[\delta_{n,0}\left(1 - \frac{\eta_i}{2}k^2\right) + \delta_{n,1}\Theta + \delta_{n,2}\frac{\eta_i}{2}\right], \end{aligned} \tag{3}$$

where $\delta_{n,m} = 1$ for $n = m$ and 0 for $n \neq m$. The phase mixing process associated with the parallel streaming of particles are now represented by the interaction to the adjacent-order [$(n-1)$ and $(n+1)$] terms in the Hermite-polynomial expansion of the perturbed distribution function with the same wave number vector $\boldsymbol{k}$ as shown in the second and third terms on the left-hand side of eq. (3). On the other hand, the $\boldsymbol{E} \times \boldsymbol{B}$ convection, which is given by the last term on the left-hand side, involves the distribution functions of only the same order n but with different wave number vectors $\boldsymbol{k}''$. The linear source terms proportional to $\Psi_{\boldsymbol{k}}$ on the right-hand side disappear for $n \geq 3$, which is the reason why the Hermite-polynomial expansion is employed here. A clear cutoff of the source like this never occurs if we use the Fourier expansion in terms of $\exp(\mathrm{i}lv_\parallel)$ $(-\infty < l < \infty)$ as basis functions.

We define the entropy variable δS by [4-8]

$$\begin{aligned} \delta S &\equiv \sum_{\boldsymbol{k}} \int \mathrm{d}v_\parallel \frac{\left\langle |\tilde{f}_{\boldsymbol{k}}|^2 \right\rangle}{2F_M} \equiv \sum_n \sum_{\boldsymbol{k}} \frac{1}{2} n! \left\langle |\hat{f}_{\boldsymbol{k},n}|^2 \right\rangle \\ &\equiv \sum_n \sum_{\boldsymbol{k}} \delta S_{\boldsymbol{k},n} \equiv \sum_n \delta S_n, \end{aligned} \tag{4}$$

where $\delta S_{\boldsymbol{k},n} \equiv \frac{1}{2} n!\, \langle |\hat{f}_{\boldsymbol{k},n}|^2 \rangle$ and $\delta S_n \equiv \sum_{\boldsymbol{k}} \delta S_{\boldsymbol{k},n}$ represent the entropy spectral functions in the $(\boldsymbol{k},n)$-space and in the n-space, respectively, and $\langle \cdots \rangle$ denotes the ensemble average. Using eq. (1), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta S_n = \eta_i Q_i \delta_{n,2} + J_{n-1/2} - J_{n+1/2} - 2\nu n \delta S_n \quad \text{for} \quad n \geq 2, \tag{5}$$

where

$$J_{n-1/2} \equiv \sum_{\boldsymbol{k}} \Theta k_y n!\ \mathrm{Im}\left\langle \hat{f}_{\boldsymbol{k},n-1} \hat{f}^*_{\boldsymbol{k},n} \right\rangle, \tag{6}$$

$$J_{n+1/2} \equiv \sum_{\boldsymbol{k}} \Theta k_y (n+1)!\ \mathrm{Im}\left\langle \hat{f}_{\boldsymbol{k},n} \hat{f}^*_{\boldsymbol{k},n+1} \right\rangle, \tag{7}$$

$$Q_i \equiv \frac{1}{2}\sum_{\boldsymbol{k}} k_y \int \mathrm{d}v_\parallel (v_\parallel^2 - 1)\mathrm{Im}\left\langle \Psi_{\boldsymbol{k}} \tilde{f}_{-\boldsymbol{k},n=2} \right\rangle, \tag{8}$$

and $\delta_{n,m} = 1$ $(n = m)$, 0 $(n \neq m)$. Here, Q_i denotes the turbulent ion heat flux downward in the temperature gradient on the right-hand side of eq.(5), $\eta_i Q_i$ represents the entropy production supplied at $n = 2$ while $-2\nu n\delta S_n$ is the collisional entropy dissipation term. We also find that $J_{n-1/2}$ ($J_{n+1/2}$) represents the entropy transfer from the $(n-1)$th (nth) to the nth [$(n+1)$th] Hermite-polynomial portion.

Now, let us investigate the entropy spectrum δS_n in the steady turbulence, where the left-hand side of eq. (5) vanishes. Considering small scales in the velocity space ($n > 2$), we treat n as a continuous variable and use the approximation $J_{n+1/2} - J_{n-1/2} \simeq \mathrm{d}J_n/\mathrm{d}n$ to obtain $\mathrm{d}J_n/\mathrm{d}n = -2\nu n\delta S_n$. Assuming that the n-dependence of the phase of $\hat{f}_{\boldsymbol{k},n}$ is determined mainly by the phase mixing of the ballistic modes, we can put $\hat{f}_{\boldsymbol{k},n-1/2}\hat{f}^*_{\boldsymbol{k},n-1/2} \simeq i(k_y/|k_y|)\,|\hat{f}_{\boldsymbol{k},n}|^2$. Then, using eqs. (4) and (6), we have $J_n/\delta S_n \simeq 2\Theta\sqrt{n}\,\langle |k_y| \rangle_n$, where the spectral average $\langle \cdots \rangle_n \equiv (\sum_{\boldsymbol{k}}\delta S_{\boldsymbol{k},n}\cdots)/(\sum_{\boldsymbol{k}}\delta S_{\boldsymbol{k},n}) \equiv (\sum_{\boldsymbol{k}}\langle|\hat{f}_{\boldsymbol{k},n}|^2\rangle\cdots)/(\sum_{\boldsymbol{k}}\langle|\hat{f}_{\boldsymbol{k},n}|^2\rangle)$ are defined. Using the relations shown above, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(2\Theta\sqrt{n}\left\langle |k_y| \right\rangle_n \delta S_n = -2\nu\, n\, \delta S_n \quad \text{for} \quad n \geq 2. \tag{9}$$

In order to derive the functional form of δS_n, we still need to specify the n-dependence of $\langle |k_y| \rangle_n$ included in eq. (9). For this purpose, we examine the role of $\boldsymbol{E} \times \boldsymbol{B}$ convection term in eq. (3). Here, we point out the analogy of our problem to the study by Batchelor on the spectrum of the passive scalar for wave lengths smaller than the Kolmogorov scale in the large Prandtl number case [9]. Like the turbulent passive scalar in small scales, $\hat{f}_{\boldsymbol{k},n}$ for large n is considered to vary so rapidly that $\boldsymbol{E} \times \boldsymbol{B}$ flow acting on $\hat{f}_{\boldsymbol{k},n}$ is regarded as a steady one which is statistically independent of $\hat{f}_{\boldsymbol{k},n}$. Then, the strain of the steady flow causes the exponential growth of the wave number of the convected variable, $\boldsymbol{k} \propto e^{\gamma t}$ [9]. Under the phase-mixing process described by the second term in the left-hand side of eq.(1), a factor in the form of $\exp(-\mathrm{i}v_\parallel\Theta k_y/\gamma)$ is produced in the velocity distribution function. In the Hermite-polynomial expansion of this factor, components of order $n \simeq (\Theta k_y/\gamma)^2$ are dominant. Thus, we have the relation, $\Theta|k_y| \simeq \gamma\sqrt{n}$. Substituting this into eq. (9) and integrating it with respect to n yield the entropy spectrum δS_n,

$$\delta S_n = \frac{\sigma}{2\gamma n}\exp\left(-\frac{\nu n}{\gamma}\right), \tag{10}$$

where $\sigma \equiv 2\nu\int_0^\infty n\delta S_n \mathrm{d}n \simeq 2\nu\Sigma_n n\delta S_n$ represents the collisional entropy dissipation rate. In the steady state, the entropy dissipation and production rates balance with other so that $\sigma = \eta_i Q_i$. We find from eq. (10) that, in the range where neither entropy production nor collisional dissipation occurs ($2 < n \ll \gamma/\nu$), we expect the power-law of $\delta S_n \propto 1/n$ with $J_n = \sigma =$ const. which is analogous to the passive scalar spectrum and its power transfer in the viscous-convective subrange.

In the analytical treatment for derivation of eq. (10), $\langle|k_y|\rangle_n \propto \sqrt{n}$ increases infinitely with n. However, in numerical simulations, there exists the upper limit of $|\boldsymbol{k}|$. Even if the potential amplitude is sufficiently damped at the maximum wave number in the simulation, still $f_{\boldsymbol{k},n}$ for large $|\boldsymbol{k}|$ and large n is continuously produced by the combination of the $\boldsymbol{E}\times\boldsymbol{B}$ convection and the phase mixing process. Therefore, saturation of $\langle|k_y|\rangle_n$ with increasing n is anticipated due to the upper limit of $|\boldsymbol{k}|$. In this case, taking $\Theta\langle|k_y|\rangle_n = \gamma_M$ as independent of n and using eqs. (4) and (6), we obtain $J_n/\delta S_n = 2\,\gamma_M\sqrt{n}$ and

$$\delta S_n = \frac{\sigma}{2\gamma_M\sqrt{n}}\exp\left(-\frac{2}{3}\frac{\nu n^{3/2}}{\gamma_M}\right), \tag{11}$$

where σ is the same as given after eq. (10).

Now, let us compare the above analytical results with numerical simulation results. In our simulation, we consider the case with no zonal flow component ($k_y = 0$) [2] and use parameters $(\eta_i, \Theta) = (10, 2.5)$. In Fig. 1, the spectrum-averaged wave number $\langle|k_y|\rangle_n$ is plotted as a function of n for $(k_{\max}, \nu) = (6.4, 2\times10^{-3})$ and for $(k_{\max}, \nu) = (12.8, 1.25\times10^{-4})$. We find that $\langle|k_y|\rangle_n$ grows nearly in proportional to $\sqrt{n}$ for smaller values of n (> 2), which agrees with the estimate of $\Theta\langle|k_y|\rangle_n \simeq \gamma\sqrt{n}$ used for derivation of eq. (10). Also, saturation of $\langle|k_y|\rangle_n$ for larger n due to the upper limit $k_{\max}$ is seen in Fig. 1. The saturation starts at lower n for smaller $k_{\max}$.

From the simulation result with $k_{\max} = 6.4$ (12.8) in Fig. 1, we obtain $\gamma \simeq 0.75$ (0.5) and $\sigma \simeq 63$ (36) for the spectral function in eq. (10) while, for that in eq. (11), we put $\gamma_M = 7.5$ (15). The entropy spectra δS_n obtained by the same simulation as in Fig. 1 are compared with those given by the combination of eqs. (10) and (11) in Fig. 2, where we use eq. (10) for $n < 3\times10^2$ (10^3) and eq. (11) for $n > 3\times10^2$ (10^3) in the case of $k_{\max} = 6.4$ (12.8). In the dissipation range with large values of n, the spectrum found in the simulation is well fitted by eq. (11). Since the constant factor in eq. (10) is determined by the constraint $2\nu\int n\delta S_n \mathrm{d}n = \sigma$ with σ evaluated from the simulation result, the spectrum δS_n for lower n evaluated from eq. (10) becomes slightly smaller than that in the simulation in order to satisfy the constraint because, for higher n, the latter spectrum is smaller than the former due to the effect of finite $k_{\max}$. Thus, the entropy spectrum observed by the slab ITG turbulence simulation can be well explained by combining the analytical expressions in eqs. (10) and (11). If we can employ an sufficiently high value of $k_{\max}$, the simulation will reproduce the spectrum δS_n in eq. (10) for the whole range of $n > 2$.

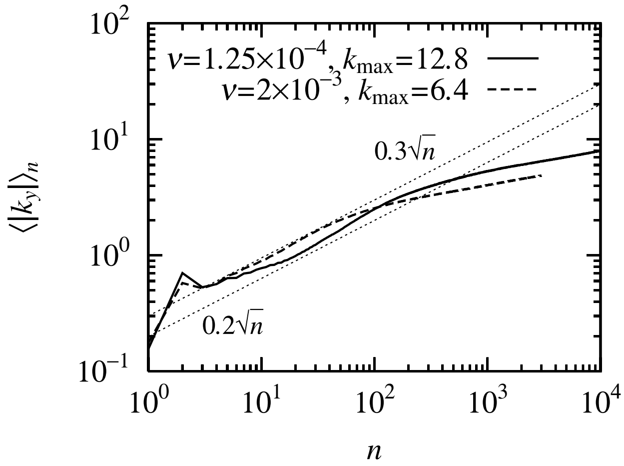

Fig. 1 Spectrum-averaged wave number $\langle|k_y|\rangle_n$ for $(k_{\max}, \nu) = (6.4, 2\times10^{-3})$ and $(k_{\max}, \nu) = (12.8, 1.25\times10^{-4})$. Upper and lower dotted straight lines represent $\langle|k_y|\rangle_n = 0.3\sqrt{n}$ and $0.2\sqrt{n}$, respectively.

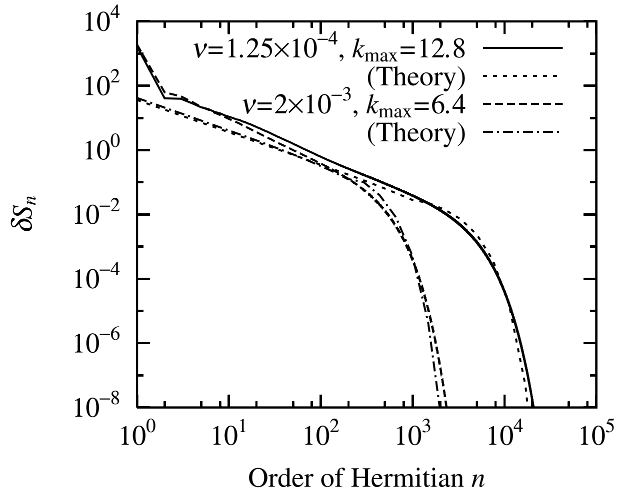

Fig. 2 Entropy spectra δS_n (solid and dashed lines) obtained by the same simulation as in Fig. 1. For comparison, dotted and dot-dashed lines represent the combination of eqs. (10) and (11). Here, eq. (10) is used for $n < 3\times10^2$ (10^3) and eq. (11) for $n > 3\times10^2$ (10^3) in the case of $k_{\max} = 6.4$ (12.8).

4. Conclusions

In this paper, the spectral analysis of the velocity distribution function in the slab ITG turbulence is made by using the Hermite-polynomial expansion with the Maxwellian weight function. The entropy variable produced at $n = 2$ by the turbulent ion heat flux downward in the temperature gradient is transfered toward the high-n side by the phase mixing process combined with the turbulent $\boldsymbol{E}\times\boldsymbol{B}$ flow and is dissipated by collisions, where n denotes the order of the

Hermite-polynomial expansion. Finding analogy to the turbulent convection of the passive scalar for wave lengths smaller than the Kolmogorov scale in the large Prandtl number case, the entropy spectrum δS_n is analytically derived and shown to well describe the numerical simulation results on how δS_n ($n > 2$) depends on n, ν (the collision frequency), and $k_{\max}$ (the maximum wave number used in the simulation). In the subrange of the n-space where neither entropy production nor collisional dissipation occurs, we obtain the power-law scaling $\delta S_n \propto 1/n$, which also resembles the form of the passive-scalar power spectrum $\propto 1/k$ in the viscous-convective wave-number subrange derived by Batchelor [9].

Acknowledgments

This work is supported in part by the Japanese Ministry of Education, Culture, Sports, Science, and Technology, Grant Nos. 12680497 and 14780387.

References

[1] W. Horton, Rev. Mod. Phys. **71**, 735 (1999).

[2] T.-H. Watanabe and H. Sugama, "*Plasma Turbulent Transport and Fine-Scale Structures of Phase Space Distribution Function*", J. Plasma Fusion Res. SERIES **6**, 23 (2004).

[3] T.-H. Watanabe, H. Sugama and T. Sato, J. Phys. Soc. Jpn. **70**, 3565 (2001).

[4] T.-H. Watanabe and H. Sugama, Phys. Plasmas **9**, 3659 (2002).

[5] H. Sugama, T.-H. Watanabe and W. Horton, Phys. Plasmas **10**, 726 (2003).

[6] H. Sugama and W. Horton, Phys. Plasmas **2**, 2989 (1995).

[7] H. Sugama, M. Okamoto, W. Horton and M. Wakatani, Phys. Plasmas **3**, 2379 (1996).

[8] J.A. Krommes and G. Hu, Phys. Plasmas **1**, 3211 (1994).

[9] G.K. Batchelor, J. Fluid Mech. **5**, 113 (1959).