## §13. Statistical Properties of the Neoclassical Radial Particle Diffusion in a Tokamak Equilibrium

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The statistical properties of the neoclassical radial particle diffusion in a tokamak equilibrium are examined by numerically evaluating the cumulant, diffusion, and autocorrelation coefficients. The gyro-phase averaged Boltzman equation in $(\vec{r}, E, \mu)$ space [ $E$ and $\mu$ are the energy and the magnetic moment, respectively ]

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f = C_T(f), \qquad (1)$$

is solved by using the Monte Carlo method. In Eq.(1), $C_T(f)$ is the linearized pitch angle scattering operator:

$$C_T(f) = \frac{\nu}{2}\frac{\partial}{\partial \lambda}\left[(1-\lambda^2)\frac{\partial f}{\partial \lambda}\right] \qquad (2)$$

where the pitch $\lambda = v_{\|}/v$ is used instead of $\mu$, and $\nu$ is the deflection collision frequency, so that the particle energy $E$ is conserved. The drift velocity $\vec{v}$ in Eq.(1) is expressed as

$$\vec{v} = v_{\|}\frac{\vec{B} + \nabla \times \left(\rho_{\|}\vec{B}\right)}{B + \hat{n}\cdot\nabla\times\left(\rho_{\|}\vec{B}\right)}, \quad \hat{n} = \frac{\vec{B}}{B}, \qquad (3)$$

with $v_{\|} = \vec{v}\cdot\hat{n}$, $\rho_{\|} = v_{\|}/\Omega$.

By using the radial particle displacement $\delta r(t) = r(t) - r(0)$, the cumulant up to 4th order, the diffusion coefficient, and the autocorrelation coefficients are defined as

$$\begin{aligned} C_1(t) &\equiv \langle \delta r(t)\rangle, \\ C_n(t) &\equiv \langle(\delta r(t) - \langle\delta r(t)\rangle)^n\rangle, \quad n = 2,3, \\ C_4(t) &\equiv \langle(\delta r(t) - \langle\delta r(t)\rangle)^4\rangle - 3C_2^2(t), \qquad (4) \\ D(t) &\equiv \frac{1}{2}\frac{dC_2(t)}{dt}, \\ A(t,t') &\equiv \frac{\langle(\delta r(t) - \langle\delta r(t)\rangle)(\delta r(t') - \langle\delta r(t')\rangle)\rangle}{\sqrt{C_2(t)C_2(t')}}, \end{aligned}$$

where $\langle\rangle$ means the particle ensemble average. For a Gaussian process, $C_{n\geq 3}(t) = 0$. The type of diffusion is distinguished as

$$\left.\begin{array}{l}\text{super}\\ \text{normal}\\ \text{sub}\end{array}\right\} \text{diffusive for } \frac{dD}{dt}\left\{\begin{array}{l} >0 \\ =0\ (D\neq 0) \\ <0 \end{array}\right.$$

and as non-diffusive for $D = 0$. When all the cumulants are invariant for time translation, and moreover $A(t,t') = A(t'-t)$, the process is statistically stationary.

The ten thousand electrons are initially loaded on a particular flux surface with fixed energy $E = 3keV$, randomly chosen pitch, and uniformly distributed poloidal and toroidal angles in the Boozer coordinates.

By analytically and numerically solving Eq.(1) without particle drift, it is found that the pitch angle scattering in the velocity space is a uniform mixing process with respect to $\lambda$ in the asymptotic time limit with $t\nu \gg 1$:

$$C_1 = 0, \quad C_2 = \frac{1}{3}, \quad \gamma_3 = 0, \ \gamma_4 = -\frac{6}{5},$$
$$D = 0, \quad A(t,t') = e^{-\nu(t'-t)}, \ t' \geq t, \qquad (5)$$

where $\gamma_n = C_n/C_2^{n/2}$. Namely, the pitch angle scattering is a uniform, non-diffusive, statistically stationary, and Markov process. This uniform mixing process in the velocity space creates the radial particle diffusion in the configuration space together with the particle drift motion. From the direct comparison of such coefficients with those of a Wiener process (classical Brownian process) with

$$C_1 = 0, \quad C_2 = 2D_{nc}t, \quad \gamma_3 = 0, \ \gamma_4 = 0,$$
$$D = D_{nc}, \quad A(t,t') = \sqrt{\frac{t}{t'}}, \quad t' \geq t, \qquad (6)$$

it is confirmed that the radial particle diffusion is a Wiener process: Gaussian, normal diffusive, statistically non-stationary, and Markov process.

In the asymptotic time limit with $\nu t \gg 1$, the pitch $\lambda$ acts as uniformly distributed independent random variable on the drift motion with a narrow radial width, so that it is considered that the radial particle diffusion becomes a Wiener process from the central limiting theorem. Note that the locality of the drift motion may lead to Fick's law:

$$\langle\vec{\Gamma}\cdot\nabla r\rangle_B = D\frac{\partial\langle n\rangle_B}{\partial r}, \qquad (7)$$

where $D$ (given by Eq.(6)) does not depend on time explicitly, and $\langle\rangle_B$ is the flux surface average, so that the equation of continuity obtained from Eq.(1) becomes well known form ($' = d/dr$):

$$\frac{\partial\langle n\rangle_B}{\partial t} = \frac{1}{V'}\frac{\partial}{\partial r}\left[V'D\frac{\partial\langle n\rangle_B}{\partial r}\right]. \qquad (8)$$