§40. A New Algorithm to Solve Saddle Points of Dispersion Relations

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To study the instability of a system, or to design the microwave oscillator driven by intensive electron beam such as backward wave oscillator (BWO), we must analyze the the condition of the absolute instability. The condition of absolute instability is given by

$$0 = \frac{\partial \omega}{\partial k} \equiv -\left(\frac{\partial D(\omega,k)}{\partial k}\right) / \left(\frac{\partial D(\omega,k)}{\partial \omega}\right),$$

where $0 = D(\omega, k)$ represent the dispersion relation of the system. Then, we need to obtain the saddle points of dispersion relation given by simultaneous equations given by

$$D(\omega_s, k_s) = 0, \text{ and } \quad \frac{\partial D(\omega_s, k_s)}{\partial k} = 0, \qquad (1)$$

in complex (k, ω) plane.

When we can derive the expressions of $\partial D/\partial k$ and $\partial D/\partial \omega$, $\partial^2 D/\partial k \partial \omega$ and $\partial^2 D/\partial k^2$ as well as $D(\omega, k)$, the simultaneous eq.(1) will be possible to be solved numerically by standard Newton iteration scheme. In actual problem such as Alfven Ion Cyclotron wave instability analysis in ICRF mirror plasma, or the design of BWO, the Dispersion relation $D(\omega, k)$ is, however, only one possible quantity to be obtained numerically.

We have derived a new computational scheme to solve the simultaneous eq.(1) using only the numerical value of dispersion relation $D(\omega, k)$ based on Newton iteration scheme.

Choosing 4 independent initial guesses (k_i, ω_i), $(i = 1, 2, 3, 4)$, close to the saddle point (k_s, ω_s), we can expand the $D(\omega, k)$ as following,

$$D_i \equiv D(k_i, \omega_i) \simeq (\omega_i - \omega_s) D_{w_s} + \frac{(k_i - k_s)^2}{2} D_{k_s k_s}$$

$$(i = 1, 2, 3, 4).$$

These equations give the expression for the improved expressions for the saddle point (k_s, ω_s) using (D_i, k_i, ω_i), $(i = 1, 2, 3, 4)$ in compact forms. Careful treatment for the reduction of the grid size generated by the 4 points (k_i, ω_i) is necessary for numerical stability, computing efficiency and for the accuracy of the solution.

A numerical example is shown in the Fig.1. Two beam instability is analyzed by this new algorithm. Dispersion relation is given by

$$D(k, \omega) = 1 - \sum_{i=0}^{1} \frac{\omega_{pi}^2}{2k^2 T_i/m_e} Z' \left(\frac{\omega - k v_i}{\sqrt{2k^2 T_i/m_e}}\right)$$

where Z' is the first derivative of the plasma dispersion function.

We have assumed that $\omega_{p1}^2/\omega_{p0}^2 = 0.2$, $T_1/T_0 = 5$, $v_1 = 10\sqrt{T_0/m_e}$. Fig.1(a) shows the frequency and wave number of the absolute instability as a function of the velocity of back ground component (0_{th} component). Fig.1(b) is typical structures of dispersion relation (ω: complex and k: real) of unstable mode.

Typical iteration number for convergence to a saddle point is 7 to 8. We can confirm the accuracy and efficiency of the new algorithm.