§18. On Linear Monte Carlo Collision Operator Conserving Momentum and Energy

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Usually, a nonlinear Monte Carlo collision operator is very time consuming. This point limits its application although it conserves momentum and energy accurately and well describes Landau collision integral. A linear Monte Carlo operator is required in many plasma simulations. Recently such linear operators are developed for gyrokinetic δf particle simulations[1,2].

The linear operators conserving momentum and energy can also serve full-f particle simulation when f is close to a Maxwellian. Consider a general Fokker-Planck equation

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \dot{Z}^i \frac{\partial f}{\partial Z^i} = C_{TP}(f) + P(\mathbf{Z})F_m \quad (1)$$

where $\mathbf{Z}$ represents the phase space variables. The test particle operator reads

$$C_{TP}(f) = -\frac{\partial}{\partial \vec{v}}[\frac{<\Delta\vec{v}>^0}{\Delta t} f] + \frac{1}{2}\frac{\partial}{\partial \vec{v}}\frac{\partial}{\partial \vec{v}} : [\frac{<\Delta\vec{v}\Delta\vec{v}>^0}{\Delta t} f] \quad (2)$$

where the fraction coefficient and diffusion tensor are calculated under Maxwellian background distribution. The source term $P(\mathbf{Z})F_m$ accounts for the momentum and energy conservation by replenishing in a Maxwellian distribution F_m the momentum and energy removed by $C_{TP}(f)$. To implement the source term one introduces changing particle weight by setting $f = w(\mathbf{Z},t)g(\mathbf{Z},t)$. In conventional particle simulation the weight is constant in time and uniform in velocity space, for example, $w(\vec{v},t) = 1$. Then, Eq.(1) is rewritten into

$$w[\frac{dg}{dt} - C_{TP}(g)] + g[\frac{dw}{dt} - C_{TP}(w) + wC_{TP}(1) - \frac{\partial w}{\partial \vec{v}}\frac{\partial \ln g}{\partial \vec{v}} : \frac{<\Delta\vec{v}\Delta\vec{v}>^0}{\Delta t}] = P(\vec{v})F_m. \quad (3)$$

Setting

$$\frac{dg}{dt} - C_{TP}(g) = 0, \quad (4)$$

we obtain the equation of evolution for the particle weight as follows

$$\frac{Dw}{Dt} \equiv \frac{dw}{dt} - C_{TP}(w) + [wC_{TP}(1) - \frac{\partial w}{\partial \vec{v}}\frac{\partial \ln g}{\partial \vec{v}} : \frac{<\Delta\vec{v}\Delta\vec{v}>^0}{\Delta t}] = \frac{P(\vec{v})F_m}{g} \quad (5)$$

Equation (4) is solved by standard particle Monte Carlo simulation to obtain $g = \Sigma_i \delta(\mathbf{Z} - \mathbf{Z}_i)$. To understand Eq.(5) and find the method to solve it, consider the Fokker-Planck Equation without the source term

$$\frac{df}{dt} = C_{TP}(f). \quad (6)$$

By the same method as for Eq.(1) we obtain

$$\frac{dg}{dt} - C_{TP}(g) = 0 \quad \text{and} \quad \frac{Dw}{Dt} = 0. \quad (7)$$

The fact the equation for g has the completely same form as Eq.(6) indicates that $w = 1$ is a solution of equation $Dw/Dt = 0$. In other words, the test particle operator C_{TP} does not change the particle weight (it changes only the phase space coordinates of particles) and Dw/Dt can be considered as the time derivation of weight along the particle trajectories as influenced by the test particle collisions. Now it is found that the particle weight can be solved by simply integrating Eq.(5) as follows ($g \approx F_m$ is assumed)

$$\Delta w|_{\mathbf{Z_i}} = \Delta t P(\mathbf{Z})|_{\mathbf{Z_i}}. \quad (8)$$

The solution for Eq.(1) is then obtained as

$$f = \Sigma_i w_i \delta(\mathbf{Z} - \mathbf{Z}_i). \quad (9)$$

Finally, we point out that in the previous work the equation for the weight is incorrect (the two terms in square bracket of Eq.(5) is lost).

Reference

1) Dimits,A.M. and Cohen,B.I. Phys. Rev. E 49 (1994) 709.
2) Lin,Z., Tang,W.M. and Lee,W.W., Plasma Phys 2 (1995) 2975.